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ON THE DIMENSION OF PRODUCT SPACES.*

By Kiiti Morita.

Let X and Y be normal spaces of finite dimension. It is well known that the relation

$$\text{(A)} \qquad \dim (X \times Y) \leqq \dim X + \dim Y$$

holds for the three cases: (a) X and Y are separable metric spaces, (b) X and Y are compact (= bicompact) spaces (E. Hemmingsen [6]) and (c) X is an S-space and Y is compact (E. G. Begle [1]). Here a Hausdorff space R is said to be an S-space or to have the star-finite property if every open covering of R has a star-finite refinement (see [1], [3], [9]), and $\dim R \leqq n$ means that every finite open covering of R has a refinement of order not greater than $n+1$.

In the present paper we shall prove the relation (A) for the following three cases:

I. The topological product of X and Y is an S-space.

II. X is a fully normal space and Y is a locally compact fully normal space.

III. X is a countably paracompact normal space and Y is a locally compact metric space.

Separable metric spaces have the Lindelöf property and the Lindelöf property implies the star-finite property for regular spaces [9], and the topological product of an S-space and a compact space is an S-space [1]. Hence the known cases mentioned above are all included in Case I. It is to be noted that a locally compact topological group, regarded as a space, has the star-finite property [9]. Our proof for Case I is based on the fact that if a subspace A of a normal space R is an S-space we have $\dim A \leqq \dim R$.

A topological space R is called paracompact or countably paracompact whenever every open covering or every countable open covering has a locally finite (= neighbourhood-finite in the sense of S. Lefschetz) refinement. It is known that, for Hausdorff spaces, paracompactness is equivalent to full normality [14] and that the topological product of a fully normal space and a

* Received September 29, 1952.

compact normal space is fully normal [2]. Recently it has been proved by C. H. Dowker [5] that (1) a topological space R is a countably paracompact normal space if and only if the topological product $R \times I$ of R with the closed line interval $I = [0, 1]$ is normal, (2) fully normal spaces and perfectly normal spaces are countably paracompact and (3) the topological product of a countably paracompact normal space and a compact metric space is countably paracompact and normal.

It will be shown that for Case II or III the product space $X \times Y$ is a paracompact or countably paracompact normal space respectively. Thus II and III are the important cases for which the topological product $X \times Y$ is normal, and a question raised by E. G. Begle [1] is solved hereby in its wider sense. Our proof of (A) for Case II rests on an addition theorem (**3**) which is a generalization of W. Hurewicz's addition theorem (cf. [10]). As an application of the relation (A) for Case III we note here that the cohomotopy group $\pi^n(X, A)$ in the sense of Borsuk-Spanier [13] can be defined for a countably paracompact normal space X with $\dim (X - A) < 2n - 1$.

The relation stronger than (A) is

$$\text{(B)} \qquad \dim (X \times Y) = \dim X + \dim Y.$$

It will be shown that (B) holds for the following cases:

IV. X is a locally compact fully normal space of dimension $\geqq 0$ and Y is a fully normal space of dimension 1.

V. X is a fully normal space of dimension $\geqq 0$ and Y is a locally finite polytope of dimension $\geqq 0$.

In case X and Y are separable metric spaces the relation (B) for Case IV was established by W. Hurewicz [7]. It follows also from our results that (B) holds for the case where X is a locally compact fully normal space of dimension $\geqq 0$ and Y is an arbitrary (finite or infinite, but non-empty) polytope assigned with the topology due to J. H. C. Whitehead [16].

1. Dimension of subspaces.

THEOREM 1. *If a subspace A of a normal space X is an S-space, then we have* $\dim A \leqq \dim X$.

Proof. Let $\{A \cap G_i \mid i = 1, 2, \cdots, s\}$ be a finite open covering of a subspace A with open sets G_i of X. For each point p of A there exists an open neighbourhood $V(p)$ whose closure is contained in some G_j. Then the family of sets $\{A \cap V(p) \mid p \varepsilon A\}$ is an open covering of A and hence, by

assumption, it has a star-finite refinement $\mathfrak{H}$. Because of star-finiteness the open covering $\mathfrak{H}$ can be written in the form $\{A \cap H_{\alpha i} \mid \alpha \varepsilon \Omega, i = 1, 2, \cdots\}$ with open sets $H_{\alpha i}$ of X such that

$$(1) \qquad A \cap H_{\alpha i} \cap H_{\beta j} = 0, \text{ for } \alpha \neq \beta.$$

By construction we have $A \cap H_{\alpha i} \subset A \cap V(p_{\alpha i})$ for some point $p_{\alpha i}$ of A, and hence we may assume without loss of generality that $H_{\alpha i} \subset V(p_{\alpha i})$. It follows then from the property of $V(p)$ that $\bar{H}_{\alpha i} \subset$ some G_j.

Now let $\dim X = n$. Then $\dim \bar{H}_{\alpha i} \leqq n$ and $\bigcup_{i=1}^{\infty} \bar{H}_{\alpha i} \subset \bigcup_{j=1}^{s} G_j$. Hence by the sum theorem (see [6], [10], [15]) there exist open sets $U_{\alpha j}$ $(j = 1, 2, \cdots, s)$ such that

$$(2) \qquad U_{\alpha j} \subset G_j, \qquad j = 1, 2, \cdots, s,$$

$$(3) \qquad \bigcup_{i=1}^{\infty} \bar{H}_{\alpha i} \subset \bigcup_{j=1}^{s} U_{\alpha j},$$

$$(4) \qquad \text{order of } \{U_{\alpha j} \mid j = 1, 2, \cdots, s\} \leqq n + 1.$$

If we put

$$W_j = \bigcup_{\alpha} W_{\alpha j}, \qquad W_{\alpha j} = U_{\alpha j} \cap (\bigcup_{i=1}^{\infty} H_{\alpha i}),$$

then $\{A \cap W_j \mid j = 1, 2, \cdots, s\}$ is an open covering of A and we have

$$(5) \qquad W_j \subset G_j, \qquad j = 1, 2, \cdots, s,$$

$$(6) \qquad \text{order of } \{A \cap W_j \mid j = 1, 2, \cdots, s\} \leqq n + 1,$$

by virtue of (1), (2) and (4). Since $\{A \cap G_j\}$ is an arbitrary open covering of A we see that $\dim A \leqq \dim X$.

Remark. It seems that the relation $\dim A \leqq \dim X$ is not true in general for a fully normal subspace A of a normal space X; for, since a Hausdorff space A of dimension zero in the sense of Menger-Urysohn is imbedded in a compact Hausdorff space X with $\dim X = 0$, the validity of the relation mentioned above implies the equivalence of dimension zero in our sense and dimension zero in the sense of Menger-Urysohn for fully normal spaces, and this equivalence is rather doubtful.

2. The relation (A) for Case I. For the sake of completeness we shall first give a simple proof to the relation (A) for the case (c); the more general Case II will be treated in **5**. Our proof given here, contrary to the existing proofs [1], [6], presupposes no properties of finite polytopes.

Let X be an S-space and Y a compact Hausdorff space, and let $\dim X \leqq m$, $\dim Y \leqq n$. Then any open covering of the product space $X \times Y$ has a refinement $\wp$ of the form

$$(7) \qquad \{G_{\alpha i} \times H(j; \alpha, i) \mid \alpha \in \Omega, i = 1, 2, \cdots ; H(j; \alpha, i) \in \mathfrak{H}_{\alpha i}\}$$

where $\{G_{\alpha i} \mid \alpha \in \Omega, i = 1, 2, \cdots\}$ is a star-finite open covering of X of order $\leqq m + 1$ such that $G_{\alpha i} \cap G_{\beta j} = 0$ for $\alpha \neq \beta$, and

$$\mathfrak{H}_{\alpha i} = \{H(j; \alpha, i) \mid j = 1, \cdots, \kappa(\alpha, i)\}$$

is a finite open covering of Y.

For each covering $\mathfrak{H}_{\alpha i}$ we construct a closed covering

$$\{K(j; \alpha, i) \mid j = 1, \cdots, \kappa(\alpha, i)\}$$

of Y such that $K(j; \alpha, i) \subset H(j; \alpha, i)$ and we apply Theorem 3.4 in our previous paper [10] to a countable number of pairs of sets

$$\{K(j; \alpha, i), H(j; \alpha, i)\}, i = 1, 2, \cdots, j = 1, 2, \cdots, \kappa(\alpha, i),$$

where α is fixed. Then we can find open sets $W(j; \alpha, i)$ of Y such that

$$(8) \qquad K(j; \alpha, i) \subset W(j; \alpha, i) \subset \overline{W(j; \alpha, i)} \subset H(j; \alpha, i),$$

$$(9) \quad \text{order of } \{\overline{W(j; \alpha, i)} - W(j; \alpha, i) \mid i = 1, 2, \cdots ; j = 1, \cdots, \kappa(\alpha, i)\} \leqq n.$$

We construct a closed covering $\{F_{\alpha i} \mid \alpha \in \Omega, i = 1, 2, \cdots\}$ of X such that $F_{\alpha i} \subset G_{\alpha i}$ and put

$$\mathfrak{M} = \{F_{\alpha i} \times \overline{V(j; \alpha, i)} \mid \alpha \in \Omega, i = 1, 2, \cdots, j = 1, 2, \cdots, \kappa(\alpha, i)\},$$

where

$$(10) \quad V(j; \alpha, i) = W(j; \alpha, i) - \bigcup_{k=1}^{j-1} \overline{W(k; \alpha, i)}, j \geqq 2; V(1; \alpha, i) = W(1; \alpha, i).$$

Then $\mathfrak{M}$ is a closed covering of $X \times Y$ and

$$F_{\alpha i} \times \overline{V(j; \alpha, i)} \subset G_{\alpha i} \times H(j; \alpha, i).$$

Hence if we prove

$$(11) \qquad \text{order of } \mathfrak{M} \leqq m + n + 1,$$

we have $\dim (X \times Y) \leqq \dim X + \dim Y$ by [11, Theorem 1.3].

Since $F_{\alpha i} \cap F_{\beta j} = 0$ for $\alpha \neq \beta$, a non-empty intersection of sets of $\mathfrak{M}$ is of the form

$$L = \bigcap_{i=1}^{r} \bigcap_{j=1}^{s_i} (F_{\alpha k_i} \times \overline{V(\tau_i(j); \alpha, k_i)}).$$

where $1 \leqq \tau_i(1) < \cdots < \tau_i(s_i) \leqq \kappa(\alpha, k_i)$, and $1 \leqq k_1 < k_2 < \cdots < k_r$. The order of the covering $\{G_{\alpha i} \mid \alpha \in \Omega, i = 1, 2, \cdots\}$ being not greater than $m+1$, we have $r \leqq m+1$. On the other hand we have by (10)

$$\bigcap_{i=1}^{r} \bigcap_{j=1}^{s_i} \overline{V(\tau_i(j); \alpha, k_i)} \subset \bigcap_{i=1}^{r} \bigcap_{j=1}^{s_i - 1} \overline{(W(\tau_i(j); \alpha, k_i) - W(\tau_i(j); \alpha, k_i))}$$

and hence we obtain $\sum_{i=1}^{r} (s_i - 1) \leqq n$ from (9). Consequently we get

$$\sum_{i=1}^{r} s_i \leqq r + n \leqq m + n + 1,$$

which proves (11).

Thus the relation (A) is established for Case (c).

Now let the topological product of X and Y be an S-space. If we denote Čech's compactifications of X and Y by $\beta(X)$ and $\beta(Y)$ respectively, then we have $\dim X = \dim \beta(X)$, $\dim Y = \dim \beta(Y)$, since by assumption X and Y are S-spaces and hence normal. The relation

$$\dim (\beta(X) \times \beta(Y)) \leqq \dim \beta(X) + \dim \beta(Y)$$

being established above, we obtain from Theorem 1

$$\dim (X \times Y) \leqq \dim (\beta(X) \times \beta(Y)) \leqq \dim X + \dim Y.$$

Hence we have

THEOREM 2. *If X and Y are spaces such that the topological product of X and Y is an S-space, then* $\dim (X \times Y) \leqq \dim X + \dim Y$.

Remark. The space S defined by Sorgenfrey [12] is an S-space with the Lindelöf property such that $S \times S$ is not normal, and moreover $\dim S = 0$. Thus the relation (A) does not hold in general unless $X \times Y$ is normal.

3. An addition theorem. We shall prove an addition theorem which plays an important role in **5**.

THEOREM 3. *Let A_α, $\alpha \in \Omega$* be closed sets of a normal space X and $\mathfrak{G}$ a locally finite open covering of X. If

1) $(\mathfrak{G}) - \dim A_\alpha \leqq n$ for every $\alpha \in \Omega$,
2) $\dim A_\alpha \cap A_\beta \leqq n - 1$ for any distinct $\alpha, \beta \in \Omega$,
3) there is a locally finite system $\{H_\alpha \mid \alpha \in \Omega\}$ of open sets of X such that $A_\alpha \subset H_\alpha$ for each α,

then we have $(\mathfrak{G})$ — $\dim \bigcup_{\alpha} A_\alpha \leqq n$. In case X is a fully normal space the condition 3) can be replaced by

3)′ $\{A_\alpha \mid \alpha \in \Omega\}$ is a locally finite system in X.

Remark. Here by $(\mathfrak{G})$ — $\dim A \leqq n$ we mean that there exists an open covering of a subspace A which is a refinement of $\mathfrak{G}$ and has order $\leqq n+1$.

Proof. The second part of the theorem follows readily from [11, Lemma, p. 22]. We assume that the set Ω of indices α consists of all transfinite ordinals less than a fixed ordinal which will be denoted by α_0, that is, we write $\{A_\alpha \mid \alpha < \alpha_0\}$, and similarly we assume that $\mathfrak{G} = \{G_\lambda \mid \lambda < \lambda_0\}$ where λ_0 is an ordinal. The proof will be carried out along the same line as that of Theorem 2. 6 in [10].

Suppose that for any ordinal β less than some ordinal $\alpha < \alpha_0$ we have constructed closed sets $P_{\beta\lambda} (\lambda < \lambda_0)$ such that

$$P_{\beta\lambda} \subset A_\beta \cap G_\lambda,$$

$$(\mathrm{T}_\beta) \qquad \bigcup_{\lambda<\lambda_0} \bigcup_{\gamma\leqq\beta} P_{\gamma\lambda} = \bigcup_{\gamma\leqq\beta} A_\gamma,$$

$$\text{order of } \{\bigcup_{\gamma\leqq\beta} P_{\gamma\lambda} \mid \lambda < \lambda_0\} \leqq n+1.$$

By the assumption 1) of the theorem such $P_{\beta\lambda}$ exist certainly for $\beta = 1$. We shall now prove the existence of $P_{\alpha\lambda}$ satisfying (T_α). If we put $Q_\lambda = \bigcup_{\beta<\alpha} P_{\beta\lambda}$, $\lambda < \lambda_0$, then Q_λ are closed, since $P_{\beta\lambda} \subset A_\beta$ and $\{A_\beta\}$ is locally finite by 3). We then have

$$(12) \qquad \text{order of } \{Q_\lambda \mid \lambda < \lambda_0\} \leqq n+1.$$

Because, if there is a point p such that $p \in Q_{\lambda_i}$ for $i = 1, 2, \cdots, n+2$, then $p \in P_{\beta_i\lambda_i}$ for some $\beta_i < \alpha$ and, since there exists β such that $\beta_i \leqq \beta < \alpha$ for every i, we have $p \in \bigcup_{\gamma\leqq\beta} P_{\gamma\lambda_i}$, for $i = 1, 2, \cdots, n+2$; this contradicts the condition (T_β).

Since $Q_\lambda \subset G_\lambda$, there can be found open sets U_λ, by [11, Theorem 1. 3], such that

$$(13) \qquad Q_\lambda \subset U_\lambda \subset G_\lambda,$$

$$(14) \qquad \text{order of } \{U_\lambda \mid \lambda < \lambda_0\} \leqq n+1.$$

On the other hand, it follows from the assumption 1) that there exist open sets V_λ and closed sets $B_\lambda (\lambda < \lambda_0)$ such that

$$(15) \qquad B_\lambda \subset V_\lambda \subset G_\lambda,$$

(16) $$\bigcup_{\lambda<\lambda_0} B_\lambda = A_\alpha,$$

(17) $$\text{order of } \{V_\lambda \mid \lambda < \lambda_0\} \leqq n+1.$$

By the assumptions 2), 3) and the generalized sum theorem [11, Theorem 3. 1] we have

(18) $$\dim A_\alpha \cap (\bigcup_{\beta<\alpha} A_\beta) \leqq n-1.$$

Since $\bigcup_{\beta<\alpha} A_\beta$ are closed, there exists a locally finite system $\{W_\tau \mid \tau < \tau_0\}$ of open sets such that

(19) $A_\alpha \cap (\bigcup_{\beta<\alpha} A_\beta) \subset \bigcup_{\tau<\tau_0} W_\tau,$

(20) $\{\bar{W}_\tau \mid \tau < \tau_0\}$ is a refinement of each of the following coverings:

$$\mathfrak{G}, \{U_\lambda, X-Q_\lambda\}, \{V_\lambda, X-B_\lambda\} \text{ for } \lambda < \lambda_0,$$

(21) order of $\{\bar{W}_\tau \mid \tau < \tau_0\} \leqq n,$

where τ_0 is an ordinal. This is assured by (18), [3, Theorem 3. 5] [11, Theorem 2. 1] and [11, Theorems 1. 2, 1. 3]. Let us put

(22) $$C = \bigcup_{\lambda<\lambda_0} C_\lambda, \qquad C_\lambda = B_\lambda \cap (X - \bigcup_{\tau<\tau_0} W_\tau)$$

and

$$M = \bigcup_{\beta<\alpha} A_\beta = \bigcup_{\lambda} Q_\lambda.$$

Then C and C_λ are closed sets and $M \cap C = 0$. Since $\{X-M, X-C\}$ can be regarded as an open covering of a normal space $\bar{W}_1$ there are closed sets E_1 and F_1 such that

(23) $$\bar{W}_1 = E_1 \cup F_1, \qquad E_1 \subset X - C, \qquad F_1 \subset X - M.$$

Then by an inductive process we can construct closed sets E_τ, F_τ $(\tau < \tau_0)$ so that

(24) $$\bar{W}_\tau = E_\tau \cup F_\tau, \qquad E_\tau \subset X - \bigcup_{\rho<\tau} (E_\rho \cap F_\rho) \cup C, \qquad F_\tau \subset X - M.$$

Because, in case we have constructed E_τ, F_τ satisfying (24) for every τ less than some ordinal $\sigma < \tau_0$, $\bigcup_{\tau<\sigma} (E_\tau \cap F_\tau)$ is closed since $\{W_\tau\}$ is locally finite, and $(\bigcup_{\tau<\sigma} (E_\tau \cap F_\tau) \cup C) \cap M = 0$, and hence the existence of E_σ, F_σ satisfying the condition analogous to (24) can be verified.

By the above construction we have

(25) $$M \cap C = 0, \qquad E_\tau \cap C = 0, \qquad F_\tau \cap M = 0 \text{ for } \tau < \tau_0,$$

then we have $(\mathfrak{G})$ — $\dim \bigcup_{\alpha} A_\alpha \leqq n$. In case X is a fully normal space the condition 3) can be replaced by

3)′ $\{A_\alpha \mid \alpha \varepsilon \Omega\}$ is a locally finite system in X.

Remark. Here by $(\mathfrak{G})$ — $\dim A \leqq n$ we mean that there exists an open covering of a subspace A which is a refinement of $\mathfrak{G}$ and has order $\leqq n + 1$.

Proof. The second part of the theorem follows readily from [11, Lemma, p. 22]. We assume that the set Ω of indices α consists of all transfinite ordinals less than a fixed ordinal which will be denoted by α_0, that is, we write $\{A_\alpha \mid \alpha < \alpha_0\}$, and similarly we assume that $\mathfrak{G} = \{G_\lambda \mid \lambda < \lambda_0\}$ where λ_0 is an ordinal. The proof will be carried out along the same line as that of Theorem 2. 6 in [10].

Suppose that for any ordinal β less than some ordinal $\alpha < \alpha_0$ we have constructed closed sets $P_{\beta\lambda} (\lambda < \lambda_0)$ such that

$$(\mathrm{T}_\beta) \qquad \begin{aligned} &P_{\beta\lambda} \subset A_\beta \cap G_\lambda, \\ &\bigcup_{\lambda<\lambda_0} \bigcup_{\gamma\leqq\beta} P_{\gamma\lambda} = \bigcup_{\gamma\leqq\beta} A_\gamma, \\ &\text{order of } \{ \bigcup_{\gamma\leqq\beta} P_{\gamma\lambda} \mid \lambda < \lambda_0\} \leqq n + 1. \end{aligned}$$

By the assumption 1) of the theorem such $P_{\beta\lambda}$ exist certainly for $\beta = 1$. We shall now prove the existence of $P_{\alpha\lambda}$ satisfying (T_α). If we put $Q_\lambda = \bigcup_{\beta<\alpha} P_{\beta\lambda}$, $\lambda < \lambda_0$, then Q_λ are closed, since $P_{\beta\lambda} \subset A_\beta$ and $\{A_\beta\}$ is locally finite by 3). We then have

$$(12) \qquad \text{order of } \{Q_\lambda \mid \lambda < \lambda_0\} \leqq n + 1.$$

Because, if there is a point p such that $p \varepsilon Q_{\lambda_i}$ for $i = 1, 2, \cdots, n + 2$, then $p \varepsilon P_{\beta_i\lambda_i}$ for some $\beta_i < \alpha$ and, since there exists β such that $\beta_i \leqq \beta < \alpha$ for every i, we have $p \varepsilon \bigcup_{\gamma\leqq\beta} P_{\gamma\lambda_i}$, for $i = 1, 2, \cdots, n + 2$; this contradicts the condition (T_β).

Since $Q_\lambda \subset G_\lambda$, there can be found open sets U_λ, by [11, Theorem 1. 3], such that

$$(13) \qquad Q_\lambda \subset U_\lambda \subset G_\lambda,$$

$$(14) \qquad \text{order of } \{U_\lambda \mid \lambda < \lambda_0\} \leqq n + 1.$$

On the other hand, it follows from the assumption 1) that there exist open sets V_λ and closed sets $B_\lambda (\lambda < \lambda_0)$ such that

$$(15) \qquad B_\lambda \subset V_\lambda \subset G_\lambda,$$

(16) $$\bigcup_{\lambda<\lambda_0} B_\lambda = A_\alpha,$$

(17) $$\text{order of } \{V_\lambda \mid \lambda < \lambda_0\} \leqq n+1.$$

By the assumptions 2), 3) and the generalized sum theorem [11, Theorem 3. 1] we have

(18) $$\dim A_\alpha \cap (\bigcup_{\beta<\alpha} A_\beta) \leqq n-1.$$

Since $\bigcup_{\beta<\alpha} A_\beta$ are closed, there exists a locally finite system $\{W_\tau \mid \tau < \tau_0\}$ of open sets such that

(19) $A_\alpha \cap (\bigcup_{\beta<\alpha} A_\beta) \subset \bigcup_{\tau<\tau_0} W_\tau,$

(20) $\{\bar{W}_\tau \mid \tau < \tau_0\}$ is a refinement of each of the following coverings:

$$\mathfrak{G}, \{U_\lambda, X - Q_\lambda\}, \{V_\lambda, X - B_\lambda\} \text{ for } \lambda < \lambda_0,$$

(21) order of $\{\ddot{W}_\tau \mid \tau < \tau_0\} \leqq n,$

where τ_0 is an ordinal. This is assured by (18), [3, Theorem 3. 5] [11, Theorem 2. 1] and [11, Theorems 1. 2, 1. 3]. Let us put

(22) $$C = \bigcup_{\lambda<\lambda_0} C_\lambda, \qquad C_\lambda = B_\lambda \cap (X - \bigcup_{\tau<\tau_0} W_\tau)$$

and

$$M = \bigcup_{\beta<\alpha} A_\beta = \bigcup_{\lambda} Q_\lambda.$$

Then C and C_λ are closed sets and $M \cap C = 0$. Since $\{X - M, X - C\}$ can be regarded as an open covering of a normal space $\bar{W}_1$ there are closed sets E_1 and F_1 such that

(23) $$\bar{W}_1 = E_1 \cup F_1, \qquad E_1 \subset X - C, \qquad F_1 \subset X - M.$$

Then by an inductive process we can construct closed sets E_τ, F_τ $(\tau < \tau_0)$ so that

(24) $$\bar{W}_\tau = E_\tau \cup F_\tau, \qquad E_\tau \subset X - \bigcup_{\rho<\tau} (E_\rho \cap F_\rho) \cup C, \qquad F_\tau \subset X - M.$$

Because, in case we have constructed E_τ, F_τ satisfying (24) for every τ less than some ordinal $\sigma < \tau_0$, $\bigcup_{\tau<\sigma} (E_\tau \cap F_\tau)$ is closed since $\{W_\tau\}$ is locally finite, and $(\bigcup_{\tau<\sigma} (E_\tau \cap F_\tau) \cup C) \cap M = 0$, and hence the existence of E_σ, F_σ satisfying the condition analogous to (24) can be verified.

By the above construction we have

(25) $$M \cap C = 0, \qquad E_\tau \cap C = 0, \qquad F_\tau \cap M = 0 \text{ for } \tau < \tau_0,$$

(26) $$(E_\rho \cap F_\rho) \cap (E_\tau \cap F_\tau) = 0 \text{ for } \rho \neq \tau.$$

We prove

(27) $$\text{order of } \{E_\tau, F_\tau \mid \tau < \tau_0\} \leqq n + 1.$$

Let

$$L = \bigcap_{i=1}^{r} E_{\rho_i} \cap (\bigcap_{j=1}^{s} F_{\tau_j}) \neq 0$$

and let us denote by $\sigma_1, \cdots, \sigma_t$ the distinct ordinals among $\rho_1, \cdots, \rho_r, \tau_1, \cdots, \tau_s$. Then we have $L \subset \bigcap_{i=1}^{t} \bar{W}_{\sigma_i}$ and hence $t \leqq n$ by (21). If there were two distinct σ_i, σ_j which are contained in $\{\rho_1, \cdots, \rho_r\}$ as well as in $\{\tau_1, \cdots, \tau_s\}$, then we would have $L \subset E_{\sigma_i} \cap F_{\sigma_i} \cap E_{\sigma_j} \cap F_{\sigma_j}$, contrary to (26). Thus we have $r + s - 1 \leqq t$ and hence $r + s \leqq t + 1 \leqq n + 1$, which proves (27).

Now let us denote by E_λ' the sum of the sets E_τ which do not intersect Q_μ for every μ less than λ and do intersect Q_λ, and by F_λ' the sum of the sets F_τ which do not intersect C_μ for any $\mu < \lambda$ and do intersect C_λ. The family of sets E_τ (or F_τ) which do not intersect M (or C) shall be denoted by $\{E_\rho'' \mid \rho < \rho_0\}$ (or $\{F_\sigma'' \mid \sigma < \sigma_0\}$), where ρ_0, σ_0 denote some ordinals. Then

(28) $$E_\rho'' \cap (M \cup C) = 0, \qquad F_\sigma'' \cap (M \cup C) = 0$$

and by (20) we have

(29) $$Q_\lambda \cup E_\lambda' \subset U_\lambda, \qquad C_\lambda \cup F_\lambda' \subset V_\lambda.$$

Let

$$\mathfrak{M} = \{Q_\lambda \cup (A_\alpha \cap E_\lambda'), C_\lambda \cup (A_\alpha \cap F_\lambda'),$$
$$A_\alpha \cap E_\rho'', A_\alpha \cap F_\sigma'' \mid \lambda < \lambda_0, \rho < \rho_0, \sigma < \sigma_0\}.$$

Then $\mathfrak{M}$ is a closed covering of $M \cup A_\alpha = \bigcup_{\gamma \leqq \alpha} A_\gamma$, since

$$\bigcup_\lambda (Q_\lambda \cup (A_\alpha \cap E_\lambda')) \cup (\bigcup_\lambda (C_\lambda \cup (A_\alpha \cap F_\lambda')))$$
$$\cup (A_\alpha \cap (\bigcup_\rho E_\rho'')) \cup (A_\alpha \cap (\bigcup_\sigma F_\sigma''))$$
$$= \bigcup_\lambda Q_\lambda \cup (\bigcup_\lambda C_\lambda) \cup (A_\alpha \cap (\bigcup_\tau E_\tau) \cup (A_\alpha \cap (\bigcup_\tau F_\tau))$$
$$= M \cup C \cup (A_\alpha \cap (\bigcup_\tau \bar{W}_\tau)) = M \cup (A_\alpha \cap (X - \cup W_\tau))$$
$$\cup (A_\alpha \cap (\cup \bar{W}_\tau)) = M \cup A_\alpha.$$

To prove

(30) $$\text{order of } \mathfrak{M} \leqq n+1,$$

suppose that

$$L = \bigcap_{i=1}^{r} [Q_{\lambda_i} \cup (A_\alpha \cap E_{\lambda_i}')] \cap [\bigcap_{j=1}^{s} (C_{\mu_j} \cup (A_\alpha \cap F_{\mu_j}'))]$$

$$\cap (\bigcap_{i=1}^{u} E_{\rho_i}'') \cap (\bigcap_{j=1}^{v} F_{\sigma_j}'') \cap A_\alpha \neq 0.$$

Then it is sufficient to prove $r+s+u+v \leqq n+1$. We distinguish three cases.

Case 1). $u \geqq 1$. By (28) we have

$$L = \bigcap_{i,j} (E_{\lambda_i}' \cap F_{\mu_j}') \cap [\bigcap_{i,j} (E_{\rho_i}'' \cap F_{\sigma_j}'')] \cap A_\alpha$$

and hence $r+s+u+v \leqq n+1$ by (27).

Case 2). $v \geqq 1$. Similarly as in Case 1).

Case 3). $u = v = 0$. In this case we have

$$L = \bigcap_{i=1}^{r} (Q_{\lambda_i} \cup (A_\alpha \cap E_{\lambda_i}')) \cap [\bigcap_{j=1}^{s} (C_{\mu_j} \cup (A_\alpha \cap F_{\mu_j}'))].$$

Case 3)$_1$. $r \geqq 1$, $s \geqq 1$. Then by (25) we have

$$L = A_\alpha \cap [\bigcap_{i,j} (E_{\lambda_i}' \cap F_{\mu_j}')] \neq 0$$

and hence $r+s \leqq n+1$ by (27).

Case 3)$_2$. $r = 0$. We have then by (29) $L \subset \bigcap_{j=1}^{s} V_{\mu_j}$ and hence $s \leqq n+1$ by (17).

Case 3)$_3$. $s = 0$. Similarly as in Case 3)$_2$.

Thus (30) is proved.

We put

$$P_{\alpha\lambda} = (A_\alpha \cap E_\lambda') \cup ((\cup' E_\rho'') \cap A_\alpha) \cup ((\cup'' F_\sigma'') \cap A_\alpha),$$

where $\cup'$ means the sum extending over all ρ such that λ is the least ordinal for which $E_\rho'' \subset G_\lambda$, and the meaning of $\cup''$ is analogous. Then $P_{\alpha\lambda}$ $(\lambda < \lambda_0)$ are closed and satisfy the condition (T_α), and the existence of $P_{\alpha\lambda}$ satisfying (T_α) is thus established for any α by transfinite induction.

Let us put finally

$$P_\lambda = \bigcup_{\alpha < \alpha_0} P_{\alpha\lambda}.$$

Then we can prove

$$\text{order of } \{P_\lambda \mid \lambda < \lambda_0\} \leqq n+1$$

similarly as in the proof of (12). Since $P_\lambda \subset G_\lambda$ and $\{P_\lambda\}$ is a closed covering of $\bigcup_\alpha A_\alpha$, we obtain $(\mathfrak{G})$ — $\dim \bigcup_{\alpha<\alpha_0} A_\alpha \leqq n$. This completes our proof.

4. Infinite polytopes. As a preparation to the following section we shall discuss some properties of infinite polytopes.

Let K be an abstract simplicial complex, finite or infinite; local finiteness is not assumed here. Let $\bar{K}$ be the polytope corresponding to this complex; the space $\bar{K}$ is a collection of closed Euclidean simplexes, each corresponding to a simplex of K (or an affine realization [8, p. 6]), and the topology of $\bar{K}$ is defined by the method of J. H. C. Whitehead [16]:

1) Each finite subpolytope has the usual Euclidean topology.

2) A subset of $\bar{K}$ is defined to be open if its intersection with every finite subpolytope $\bar{L}$ is open in the ordinary Euclidean topology of $\bar{L}$.

This topology is finer than (in general, but equivalent to in the case of locally finite complexes) the topology induced by the natural metric as well as the topology of geometric complex (cf. [8]).[1]

According to [16, Theorem 35] $\bar{K}$ is a normal space and every open covering of $\bar{K}$ has a refinement which consists of open stars $O(a, K^*)$ of all vertices of a suitable simplicial subdivision K^* of K. By [8, p. 37] the open covering $\{O(a, K^*)\}$ is point-finite and analytic. According to [4, Theorem 1 and Corollary 3], a point-finite open covering of a normal space is analytic if and only if it has a locally finite refinement (and hence it is a normal covering in the sense of J. W. Tukey). Therefore $\bar{K}$ is paracompact or equivalently fully normal. If the (combinatorial) dimension m of K, the least upper bound of dimensions of all simplexes in K, is finite, then the above consideration shows at the same time that $\dim \bar{K} \leqq \dim K$, and, since K contains an m-simplex, we have $\dim \bar{K} \geqq \dim K$ and hence $\dim \bar{K} = \dim K$. Thus

LEMMA 1. *Any polytope $\bar{K}$ is a fully normal space and the topological dimension of $\bar{K}$ coincides with the combinatorial dimension of K.*

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[1] Cf. also C. H. Dowker, "Topology of metric complexes," *American Journal of Mathematics*, vol. 74 (1952), pp. 555-577, which arrived at our university after the present work was completed.

Now let $\dim K = \dim \bar{K} = m$ be finite and let $\bar{K}'$ be a barycentric subdivision of $\bar{K}$. Here we consider $\bar{K}$ to be a polytope determined by an affine realization of K and make use of barycentric coordinates (see [8, pp. 6-8]). If we denote by $N(a, K')$ the closed star of a, that is, the sum of closed simplexes of K' which contain a, and by $\{a_\alpha \mid \alpha \varepsilon \Omega\}$ the totality of all vertices of K, then the family $\{N(a_\alpha, K')\}$ is a closed covering of $\bar{K}$. We prove

LEMMA 2. *$\{N(a_\alpha, K')\}$ is a locally finite closed covering of K in case the dimension of K is finite.*

To prove Lemma 2, let x be any point of $\bar{K}$ with the barycentric coordinates $\{x_\alpha \mid \alpha \varepsilon \Omega\}$. Then the set

$$V(x) = \{y \mid \sum_\alpha (x_\alpha - y_\alpha)^2 < 1/(m+1)^2\}$$

is easily shown to be an open neighbourhood of x. If x is contained in an open simplex $\sigma^r = (a_{\alpha_0}, a_{\alpha_1}, \cdots, a_{\alpha_r})$ of K, we have $V(x) \cap N(a_\beta, K') = 0$ for any vertex a_β which does not belong to the simplex σ^r. Because, for such β every point y of $V(x)$ has the coordinate $y_\beta < 1/(m+1)$, while for any point z of $N(a_\beta, K')$ we have $z_\beta \geqq 1/(m+1)$ as is shown by a simple calculation. This proves Lemma 2.

Let X be a fully normal space and $\mathfrak{U} = \{U_\alpha \mid \alpha \varepsilon \Omega\}$ a locally finite open covering of X. Then there exists an open covering $\{V_\alpha \mid \alpha \varepsilon \Omega\}$ such that $\bar{V}_\alpha \subset U_\alpha$ for every α. For each α there exists a non-negative bounded continuous function $f_\alpha(x)$ of X such that $f_\alpha(x) = 1$ for $x \varepsilon \bar{V}_\alpha$ and $f_\alpha(x) = 0$ for $x \varepsilon X - U_\alpha$. Let $N(\mathfrak{U})$ be the nerve of $\mathfrak{U}$ and let us denote by u_α the vertex of $N(\mathfrak{U})$ corresponding to U_α of $\mathfrak{U}$. Then a mapping ϕ of X into the polytope $\overline{N(\mathfrak{U})}$ defined by

$$\phi\colon x \to y = \{y_\alpha\}, \qquad y_\alpha = f_\alpha(x) / \sum_\beta f_\beta(x),$$

where $\{y_\alpha\}$ means the barycentric coordinates of a point y of $\overline{N(\mathfrak{U})}$, is continuous. Because for a point x_0 of X there exists an open neighbourhood $V(x_0)$ such that $V(x_0)$ meets only a finite number of sets of $\mathfrak{U}$; these sets will be denoted by U_{α_i}, $i = 0, 1, \cdots, r$. Then ϕ maps $V(x_0)$ into a closed simplex $\bar{\sigma}^r$ of $N(\mathfrak{U})$ whose vertices are u_{α_i}, $i = 0, 1, \cdots, r$. Since a partial mapping $\phi \mid V(x_0) : V(x_0) \to \bar{\sigma}^r$ is evidently continuous, ϕ itself is continuous. It is easy to see that $\phi^{-1}(O(u_\alpha, N(\mathfrak{U}))) \subset U_\alpha$ for each α, that is, ϕ is a canonical mapping with respect to $\mathfrak{U}$.

5. The relation (A) for Case II.

THEOREM 4. *Let X be a fully normal space and Y a locally compact fully normal space. Then the topological product of X and Y is fully normal, and*

$$\dim (X \times Y) \leqq \dim X + \dim Y. \tag{A}$$

Proof. 1) First we shall prove (A) for the case where Y is compact. In case $\dim X = 0$, X is an S-space and (A) holds for this case by Theorem 2; but in this case the covering $\mathfrak{P}$ defined below (see (31)) has order $\leqq n+1$ and thus we have a direct proof. Suppose that (A) has been proved for at most $(m-1)$-dimensional X. Under this assumption of induction we shall prove (A) for an m-dimensional X. We let $\dim Y = n$.

Any open covering $\mathfrak{G}$ of the product space $X \times Y$ has, as is well known, a refinement $\mathfrak{P}$ of the form

$$\{U_\alpha \times W_{\alpha_i} \mid \alpha \in \Omega, W_{\alpha_i} \in \mathfrak{W}_\alpha\} \tag{31}$$

where $\mathfrak{U} = \{U_\alpha \mid \alpha \in \Omega\}$ is a locally finite open covering of X of order $\leqq m+1$ and $\mathfrak{W}_\alpha$ is a finite open covering of Y of order $\leqq n+1$ for each α (cf. [3, p. 220]). Let $N(\mathfrak{U})$ be the nerve of $\mathfrak{U}$ and let us make use of the same notations concerning $N(\mathfrak{U})$ as in the preceding section **4**. We put further $K = N(\mathfrak{U})$. As is shown in **4** there is a continuous mapping ϕ of X into $\bar{K}$ such that $\phi^{-1}(O(u_\alpha, K)) \subset U_\alpha$ for every α. Then a mapping Φ of $X \times Y$ into $\bar{K} \times Y$ defined by $\Phi(x, y) = (\phi(x), y)$, for $x \in X$, $y \in Y$ is clearly a continuous mapping and

$$\Phi^{-1}(O(u_\alpha, K) \times W_{\alpha_i}) \subset U_\alpha \times W_{\alpha_i}. \tag{32}$$

If we denote by $\bar{K}'$ the barycentric subdivision of $\bar{K}$, then by Lemma 2 in **4** $\{N(u_\alpha, K') \mid \alpha \in K\}$ is a locally finite closed covering of $\bar{K}$. Hence, since $N(u_\alpha, K') \subset O(u_\alpha, K)$, there is a locally finite open covering $\{H_\alpha \mid \alpha \in \Omega\}$ of $\bar{K}$ such that $N(u_\alpha, K') \subset H_\alpha \subset O(u_\alpha, K)$, by virtue of Lemma 1 and [11, Lemma, p. 22]. Then the family

$$\mathfrak{M} = \{H_\alpha \times W_{\alpha_i} \mid \alpha \in \Omega, W_{\alpha_i} \in \mathfrak{W}_\alpha\}$$

is a locally finite open covering of $\bar{K} \times Y$. Since $N(u_\alpha, K') \subset H_\alpha$ and $\mathfrak{W}_\alpha$ has order $\leqq n+1$, we have

$$(\mathfrak{M}) - \dim [N(u_\alpha, K') \times Y] \leqq n. \tag{33}$$

On the other hand, in case $\alpha \neq \beta$ the intersection $N(u_\alpha, K') \cap N(u_\beta, K')$

is a sum of simplexes of dimension $\leqq m-1$. Hence by Lemma 1 we have $\dim N(u_\alpha, K') \cap N(u_\beta, K') \leqq m-1$ and consequently by the assumption of induction we get

(34) $\dim [(N(u_\alpha, K') \times Y) \cap (N(u_\beta, K') \times Y)] \leqq m+n-1.$

Therefore, since $\bar{K} \times Y$ is fully normal, we obtain from (33), (34) and Theorem 3

$$(\mathfrak{M}) - \dim \bar{K} \times Y \leqq m+n.$$

Thus there exists an open covering $\{V_\gamma\}$ of $\bar{K} \times Y$ which is a refinement of $\mathfrak{M}$ and has order $\leqq m+n+1$. Then $\{\Phi^{-1}(V_\gamma)\}$ is clearly a refinement of $\wp$ in view of (32) and has order $\leqq m+n+1$. Hence we have $(\mathfrak{G}) - \dim (X \times Y) \leqq m+n$. Consequently the relation (A) is established by induction for the case where Y is compact.

2) Let Y be a locally compact fully normal space. Then there exists a star-finite open covering $\{G_\gamma \mid \gamma \varepsilon \Gamma\}$ of Y such that the closure of each G_γ is compact. Let $\mathfrak{U}$ be any open covering of $X \times Y$. Then for each γ there is a locally finite open covering $\mathfrak{U}_\gamma$ of $X \times \bar{G}_\gamma$ which is a refinement of $\mathfrak{U}$. If we denote by $\mathfrak{B}_\gamma$ the family $\{(X \times G_\gamma) \cap U \mid U \varepsilon \mathfrak{U}_\gamma\}$ it is easy to see that the collection $\mathfrak{B}$ of all $\mathfrak{B}_\gamma$: $\mathfrak{B} = \{V \mid V \varepsilon \mathfrak{B}_\gamma, \gamma \varepsilon \Gamma\}$ is a locally finite open covering of $X \times Y$. Since $\mathfrak{B}$ is clearly a refinement of $\mathfrak{U}$, this proves that the product space $X \times Y$ is fully normal.

The covering $\{X \times \bar{G}_\gamma \mid \gamma \varepsilon \Gamma\}$ is locally finite and by the proof given in 1) we have

$$\dim (X \times \bar{G}_\gamma) \leqq \dim X + \dim \bar{G}_\gamma \leqq \dim X + \dim Y.$$

Hence from the generalized sum theorem [11, Theorem 3. 2] it follows that $\dim (X \times Y) \leqq \dim X + \dim Y$.

Thus the theorem is completely proved.

6. The relation (A) for Case III.

THEOREM 5. *If X is a countably paracompact normal space and Y a locally compact metric space, then the topological product $X \times Y$ of X with Y is a countably paracompact normal space and the relation* $\dim (X \times Y) \leqq \dim X + \dim Y$ *holds.*

Proof. 1) First we shall deal with the case where Y is compact. Let $\dim X = m$, $\dim Y = n$ and $s = m+n+1$. Let $P^{(i)}, Q^{(i)}$ $(i = 1, 2, \cdots, s)$ be s pairs of closed sets of the product space $X \times Y$ such that $P^{(i)} \cap Q^{(i)} = 0$

for $i=1,2,\cdots,s$. We shall prove that there exist open sets $U^{(i)}$. $i=1,\cdots,s$ of $X\times Y$ such that

$$P^{(i)}\subset U^{(i)}\subset \bar{U}^{(i)}\subset X\times Y-Q^{(i)},\qquad i=1,2,\cdots,s, \tag{35}$$

$$\bigcap_{i=1}^{s} B(U^{(i)})=0. \tag{36}$$

Here $B(A)$ means the boundary of a set A.

Since Y is a compact metric space, there is a countable basis for open sets of Y. We take s arbitrary countable bases $\mathfrak{L}^{(i)}$, $i=1,2,\cdots,s$ of Y and denote by $\{M_j^{(i)},L_j^{(i)}\}$, $j=1,2,\cdots$ the totality of pairs of sets of $\mathfrak{L}^{(i)}$ such that $\bar{M}_j^{(i)}\subset L_j^{(i)}$. We now apply our Theorem 3.4 in [10] to

$$\{\bar{M}_j^{(i)},L_j^{(i)}\},\qquad i=1,2,\cdots,s,\ j=1,2,\cdots.$$

Then we can find open sets $G_j^{(i)}$ such that $\bar{M}_j^{(i)}\subset G_j^{(i)}\subset L_j^{(i)}$ and

$$\text{order of the family } \{B(G_j^{(i)})\mid i=1,2,\cdots,s;\, j=1,2,\cdots\}\leqq n. \tag{37}$$

We note that $\{G_j^{(i)}\mid j=1,2,\cdots\}$ is a countable basis of Y for each i.

Denoting by Γ the totality of all finite subsets of the set of natural numbers, we put $H_\gamma^{(i)}=\bigcup_{j\varepsilon\gamma}G_j^{(i)}$, for $\gamma\,\varepsilon\,\Gamma$. For each point x of X let $P_x^{(i)}$ be the closed set of Y defined by $x\times P_x^{(i)}=(x\times Y)\cap P^{(i)}$; similarly let $x\times Q_x^{(i)}=(x\times Y)\cap Q^{(i)}$. Let us put further

$$V_\gamma^{(i)}=\{x\mid P_x^{(i)}\subset H_\gamma^{(i)}\subset \bar{H}_\gamma^{(i)}\subset Y-Q_x^{(i)}\}. \tag{38}$$

Then $\{V_\gamma^{(i)}\mid \gamma\,\varepsilon\,\Gamma\}$ is an open covering of X as is shown by C. H. Dowker [5, p. 222]. Since by assumption X is countably paracompact and the cardinal number of Γ is countable, there is a locally finite countable open covering $\{W_\gamma^{(i)}\mid\gamma\,\varepsilon\,\Gamma\}$ of X such that $W_\gamma^{(i)}\subset V_\gamma^{(i)}$ for $\gamma\,\varepsilon\,\Gamma$. It follows further that there is a closed covering $\{F_\gamma^{(i)}\mid\gamma\,\varepsilon\,\Gamma\}$ such that $F_\gamma^{(i)}\subset W_\gamma^{(i)}$. We now apply our Theorem 3.4 in [10] again to

$$\{F_\gamma^{(i)},W_\gamma^{(i)}\},\qquad i=1,2,\cdots,s;\,\gamma\,\varepsilon\,\Gamma.$$

Then we can find open sets $U_\gamma^{(i)}$ of X such that

$$F_\gamma^{(i)}\subset U_\gamma^{(i)}\subset W_\gamma^{(i)}, \tag{39}$$

$$\text{order of } \{B(U_\gamma^{(i)})\mid i=1,2,\cdots,s;\,\gamma\,\varepsilon\,\Gamma\}\leqq m. \tag{40}$$

Now let us put $U^{(i)}=\bigcup_{\gamma\varepsilon\Gamma}(U_\gamma^{(i)}\times H_\gamma^{(i)})$. Since $\{U_\gamma^{(i)}\times H_\gamma^{(i)}\mid\gamma\,\varepsilon\,\Gamma\}$ is locally finite, we have $\bar{U}^{(i)}=\bigcup_\gamma\overline{U_\gamma^{(i)}\times H_\gamma^{(i)}}$ and hence

(41) $$B(U^{(i)}) \subset \bigcup_{\gamma \varepsilon \Gamma} B(U_\gamma^{(i)} \times H_\gamma^{(i)}).$$

For any point (x, y) of $P^{(i)}$ we have $x \varepsilon U_\gamma^{(i)}$ for some $\gamma \varepsilon \Gamma$ and hence $(x, y) \varepsilon U_\gamma^{(i)} \times H_\gamma^{(i)}$, and consequently

(42) $$P^{(i)} \subset U^{(i)}.$$

On the other hand, $\overline{U_\gamma^{(i)} \times H_\gamma^{(i)}} \cap Q^{(i)} = (\bar{U}_\gamma^{(i)} \times \bar{H}_\gamma^{(i)}) \cap Q^{(i)} = 0$, and hence

(43) $$\bar{U}^{(i)} \subset X \times Y - Q^{(i)}.$$

We now prove

(44) $$\bigcap_{i=1}^{s} B(U^{(i)}) = 0.$$

For this purpose it is sufficient, in view of (41), to prove that

$$L = \bigcap_{i=1}^{s} B(U_{\gamma_i}^{(i)} \times H_{\gamma_i}^{(i)}) = 0, \text{ for } \gamma_i \varepsilon \Gamma.$$

Denoting by Δ the family of all subsets δ of $\{1, 2, \cdots, s\}$ we have

$$L = \bigcup_{\delta \varepsilon \Delta} (E_\delta \cap F_\delta),$$

where

$$E_\delta = \bigcap_{i \varepsilon \delta} (B(U_{\gamma_i}^{(i)}) \times \bar{H}_{\gamma_i}^{(i)}), \qquad F_\delta = \bigcap_{i \varepsilon \bar{\delta}} (\bar{U}_{\gamma_i}^{(i)} \times B(H_{\gamma_i}^{(i)})).$$

In case the cardinal number of δ is greater than or equal to $m + 1$, we see by (40) that $E_\delta = 0$, and in case the cardinal number of δ is less than $m + 1$, we have $F_\delta = 0$ by (37) since $B(H_\gamma^{(i)}) \subset \bigcup_{j \varepsilon \gamma} B(G_j^{(i)})$. Hence we have $E_\delta \cap F_\delta = 0$ in every case and consequently $L = 0$. This proves (44).

Thus the existence of open sets $U^{(i)}$ satisfying (35), (36) is established, and hence, according to a generalization of Eilenberg-Otto's theorem (see [6], [10]) we see that $\dim (X \times Y) \leqq \dim X + \dim Y$ for a compact metric space Y.

2) Let Y be a locally compact metric space. Then there exists a star-finite open covering $\{G_\alpha \mid \alpha \varepsilon \Omega\}$ of Y such that the closure of each G_α is compact [9]. Similarly as in 2) of the proof of Theorem 4 we can prove that the product space $X \times Y$ is countably paracompact.

We next prove the normality of $X \times Y$. Let P, Q be disjoint closed sets of $X \times Y$. Since $X \times \bar{G}_\alpha$ is normal [5], there exists an open set H_α of $X \times \bar{G}_\alpha$ such that $P \cap (X \times \bar{G}_\alpha) \subset H_\alpha$, $\bar{H}_\alpha \cap Q \cap (X \times \bar{G}_\alpha) = 0$. If we put $K_\alpha = H_\alpha \cap (X \times G_\alpha)$, then K_α is an open set of $X \times Y$ and we have

$P \cap (X \times G_\alpha) \subset K_\alpha$, $\bar{K}_\alpha \cap Q = 0$, and hence $P \subset K$, $\bar{K} \cap Q = 0$ where $K = \bigcup_\alpha K_\alpha$, since $\{K_\alpha\}$ is locally finite. This proves that $X \times Y$ is normal.

Finally we construct a closed covering $\{F_\alpha \mid \alpha \varepsilon \Omega\}$ of Y such that $F_\alpha \subset G_\alpha$. Then, as is proved in 1), we have

$$\dim (X \times F_\alpha) \leqq \dim X + \dim F_\alpha \leqq \dim X + \dim Y.$$

Since $X \times F_\alpha \subset X \times G_\alpha$ and $\{X \times G_\alpha\}$ is locally finite, we have $\dim (X \times Y) \leqq \dim X + \dim Y$ by virtue of the generalized sum theorem [11, Theorem 3.1]. Thus Theorem 5 is completely proved.

7. The relation (B) for Cases IV and V. Let X be a fully normal space and let $\dim X = n$ (n finite). In the following we assume $n \geqq 1$ and apply Hurewicz's method [7]; in case $n = 0$ the relation (B) is clearly true. Since $\dim X = n$, there exist a closed set A and a continuous mapping f of A into an $(n-1)$-sphere S^{n-1} such that f is not extensible over X (see [3], [6], [10]). Assuming that S^{n-1} is a boundary of an n-simplex σ^n we extend f to a continuous mapping F from X into $\bar{\sigma}^n$. There is no loss of generality in assuming that $A = F^{-1}(S^{n-1})$. We denote by X_0 the space obtained from X by contracting the closed set A to a point p_0 (that is, X_0 is the decomposition space determined by the decomposition $X = \bigcup_{x \varepsilon X - A} x \cup A$) and by S_0^n the space obtained from $\bar{\sigma}^n$ by contracting S^{n-1} to a point q_0; the continuous mappings associated with these decompositions will be denoted by $\phi : X \to X_0$; $\psi : \bar{\sigma}^n \to S_0^n$ respectively. S_0^n is clearly homeomorphic to an n-sphere and X_0 is fully normal. If we define a continuous mapping f_0 of X_0 into S_0^n by $f_0(x) = \psi(F(\phi^{-1}(x)))$ for $x \varepsilon X_0 - p_0$, $f_0(p_0) = q_0$, then f_0 is an essential mapping of X_0 into S_0^n. This result is established by C. H. Dowker [3, p. 235].

We observe that $\dim X_0 = n$. Since $A = F^{-1}(S^{n-1})$, A is a G_δ-set of X and hence there exist a countable number of closed sets A_i, $i = 1, 2, \cdots$ such that

$$X - A = \bigcup_{i=1}^{\infty} A_i. \tag{45}$$

Since $\dim A_i \leqq n$ for every i and

$$X_0 - p_0 = \bigcup_{i=1}^{\infty} A_i, \tag{46}$$

we have $\dim X_0 \leqq n$ by the sum theorem (see [6], [10], [15]). On the other hand, f_0 is an essential mapping of X_0 into S_0^n and hence $\dim X_0 \geqq n$. Thus we have $\dim X_0 = n$.

Now we prove

$$\dim (X_0 \times Y) \geqq n + 1 \tag{47}$$

for the following cases:

IV. X is compact and Y is a fully normal space of dimension $\geqq 1$.

V. X is fully normal and Y is the closed line interval $I = [0, 1]$.

Case IV. In this case X_0 is compact. Suppose, contrary to (47), that $\dim (X_0 \times Y) \leqq n$. Since $\dim Y \geqq 1$, there exist disjoint closed sets P, Q such that $\bar{V} - V \neq 0$ for any open set V satisfying $P \subset V \subset Y - Q$. Let us define a continuous mapping Φ_0 from $X_0 \times (P \cup Q)$ into $S_0{}^n$ by

$$\Phi_0(x, y) = f_0(x) \quad \text{for} \quad x \,\varepsilon\, X_0, y \,\varepsilon\, P,$$

$$\Phi_0(x, y) = q_0 \qquad \text{for} \quad x \,\varepsilon\, X_0, y \,\varepsilon\, Q.$$

Since $\dim (X_0 \times Y) \leqq n$, Φ_0 can be extended to a continuous mapping Φ from $X_0 \times Y$ into $S_0{}^n$ by a well-known theorem ([3], [6], [10]). Let us put

$$\Phi_y(x) = \Phi(x, y)$$

and denote by V the set of points y of Y such that $\Phi_y : X_0 \to S_0{}^n$ is an essential mapping. Then we have obviously $P \subset V \subset Y - Q$. For each point y of Y there is an open neighbourhood $U(y)$ such that Φ_y and $\Phi_{y'}$ are homotopic for any point y' of $U(y)$, since X_0 is compact. Therefore V and $Y - V$ are open sets, that is, $\bar{V} - V = 0$. Thus we have arrived at a contradiction and (47) is established.

Case V. X_0 is fully normal and Y is the closed line interval I. Suppose that $\dim (X_0 \times I) \leqq n$. Then a continuous mapping Φ_0 of $(X_0 \times 0) \cup (X_0 \times 1)$ into $S_0{}^n$ defined by

$$\Phi_0(x, 0) = f_0(x) \quad \text{for} \quad x \,\varepsilon\, X_0$$

$$\Phi_0(x, 1) = q_0 \qquad \text{for} \quad x \,\varepsilon\, X_0$$

can be extended to a continuous mapping Φ of $X_0 \times Y$ into $S_0{}^n$. But the existence of such a mapping Φ means that the mapping f_0 is homotopic (in the ordinary sense; see [3, p. 204]) to a constant mapping. This contradicts the fact that f_0 is essential. This proves (47).

The relation (47) is thus established for both cases. The product space $X \times Y$ is normal and in view of (46)

$$X_0 \times Y = (p_0 \times Y) \cup (\bigcup_{i=1}^{\infty} (A_i \times Y)).$$

Since $p_0 \times Y$ is homeomorphic to a closed subset of $A_i \times Y$ for any non-empty A_i, by virtue of the sum theorem there exists some A_j $(j \geqq 1)$ such that $\dim (X_0 \times Y) = \dim (A_j \times Y)$. Hence we have, for Cases IV and V, $\dim (X \times Y) \geqq \dim (A_j \times Y) = \dim (X_0 \times Y) \geqq n + 1$.

In case X is a locally compact fully normal space, there is a locally finite closed covering $\{F_\alpha\}$ of X such that F_α is compact, and hence by the generalized sum theorem [11, Theorem 3.2] we have $\dim X = \dim F_\alpha = n$ for some α. Therefore

$$\dim (X \times Y) \geqq \dim (F_\alpha \times Y) \geqq n + 1,$$

where Y is a fully normal space of dimension $\geqq 1$. Thus we obtain

THEOREM 6. *If X is a locally compact fully normal space of dimension n $(n \geqq 0)$ and Y a fully normal space of dimension $\geqq 1$, then* $\dim (X \times Y) \geqq n + 1$.

Our Theorem 6 clearly establishes the relation (B) for Case IV (see Introduction) in view of Theorem 4.

The discussion for Case V, together with Theorem 4, leads to the following theorem.

THEOREM 7. *If X is a fully normal space of dimension $\geqq 0$ and Y is a locally finite polytope of dimension $\geqq 0$, then*

$$\dim (X \times Y) = \dim X + \dim Y.$$

Finally we obtain from Theorems 4, 7 and Lemma 1:

THEOREM 8. *If X is a locally compact fully normal space of dimension $\geqq 0$ and Y is an arbitrary (finite or infinite) polytope of dimension $\geqq 0$ (see* **4**), *then we have* $\dim (X \times Y) = \dim X + \dim Y$.

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COMPACTNESS CONDITIONS AND UNIFORM STRUCTURES.*

By Alice Dickinson.

In a completely regular topological space there exists at least one uniform structure compatible with the topology of the space. In this paper, relations between certain compactness conditions on a space and the set of compatible uniform structures on a space are considered. The terminology follows closely that of Bourbaki [2].

A uniform structure with filter $\mathfrak{F}_1$ is finer than a uniform structure with filter $\mathfrak{F}_2$ if $\mathfrak{F}_1$ is finer than $\mathfrak{F}_2$. With respect to this relation the uniform structures on a space form a partially ordered set. Weil [9], p. 16, showed that there is always one compatible uniform structure finer than all the others. This upper bound on the set of uniform structures is called the universal uniform structure of the space.

Consider the set of uniform structures compatible with the topology of a given space. Let a uniform structure in this set, less fine than all the other elements in the set, be called the *crude* uniform structure of the space.

Theorem 1. *Let E be a locally compact space. Then the uniform structure induced by the uniquely defined compactification $E^* = E \cup \xi$ is the crude uniform structure of the space.*

Proof. Alexandroff and Urysohn [1], p. 68, proved that a locally compact space can be compactified by the addition of a single point in a unique manner. Let $\mathfrak{W}$ be a symmetric fundamental system of entourages of the uniform structure induced by this compact space, $E \cup \xi$. Let $\mathfrak{V}$ be a symmetric system of entourages of an arbitrary uniform structure compatible with the topology of E. Suppose the filter of $\mathfrak{V}$ is not finer than that of $\mathfrak{W}$. Then there exists an entourage W such that for every V_α in $\mathfrak{V}$, the set $V_\alpha - W$ is not empty. Let $F_\alpha = V_\alpha - W$. Then the sets $\{F_\alpha\}$ form the basis of a filter on $E \times E$. The filter basis $\{F_\alpha\}$ has no contiguous point in $E \times E$,

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since any such point would be contained both in $\bigcap_{\alpha} \bar{V}_\alpha = \Delta$ and in the complement of W; but these are disjoint sets. However, the filter basis $\{F_\alpha\}$ does have a contiguous point in the compact space $E^* \times E^*$. There are two cases to consider. First, suppose (ξ, ξ) is contiguous to the filter basis $\{F_\alpha\}$. The entourage W is the trace of an entourage W^* in $E^* \times E^*$. The entourage W^* contains an open set containing (ξ, ξ), and hence intersects every F_α. It follows that W intersects every F_α, which is contrary to the definition of the F_α. On the other hand, suppose the point (ξ, x_0) is contiguous to $\{F_\alpha\}$, and thus also (x_0, ξ) since the fundamental systems are symmetric. In E^* there exist disjoint neighborhoods $N_1(x_0)$ and $N_2(\xi)$. Since the uniform structure defined by $\mathfrak{V}$ is compatible with the topology of E, there is an entourage V_γ such that $V_\gamma(x_0) \subset N_1(x_0)$. And by the uniform structure axioms there is an entourage V_β such that $V_\beta \circ V_\beta \subset V_\gamma$. Let (r, s) be a point in $V_\beta \cap [V_\beta(x_0) \times N_2(\xi)]$ which is not empty since (x_0, ξ) is contiguous to $\{F_\alpha\}$. It follows that (x_0, r) is in V_β since r is in $V_\beta(x_0)$. Thus $(x_0, s) \in V_\beta \circ V_\beta \subset V_\gamma$ which implies that $s \in V_\gamma(x_0)$. Hence $V_\gamma(x_0) \cap N_2(\xi)$ is not empty. This is a contradiction.

Conversely, if the space E is not locally compact, then no uniform structure induced by a compactification is a crude uniform structure. In a space which is not locally compact, any compactification requires the addition of an infinite number of points. The identification of any two of these added points gives a compactification which induces a uniform structure which is less fine than that induced by the original compactification. This statement will be demonstrated by the construction in the proof of Theorem 2.

THEOREM 2. *If a space E has a unique uniform structure, then it has a unique compactification. This compactification consists of the addition of a single point. The unique uniform structure is, of course, induced by the compactification.*

Proof. Let $E^\dagger$ be a compactification obtained by the addition of points including $y_1 \neq y_2$. Let $\mathfrak{V}$ be a symmetric system of entourages defining the compatible uniform structure on $E^\dagger$. Let $\mathfrak{V}' = \{V'_\alpha\}$ be a new fundamental system of entourages, where

$$V'_\alpha = V_\alpha \cup [V_\alpha(y_1) \times V_\alpha(y_2)] \cup [V_\alpha(y_2) \times V_\alpha(y_1)]$$

for all $V_\alpha \in \mathfrak{V}$. Each V'_α is symmetric and contains $\Delta^\dagger$. Thus the satisfaction of the uniform structure axioms may be established by showing that

$V_\beta \circ V_\beta \subset V_\alpha$ implies $V'_\beta \circ V'_\beta \subset V'_\alpha$. If the points (a, b) and (b, d) are in V'_β, then one of the following combinations of elements, where $i \neq j$ take on the values 1 and 2, is in V_β: (1) $(a, b), (b, d)$; (2) $(y_i, a), (y_j, b), (y_i, d)$; (3) $(y_i, a), (y_j, b), (b, d)$; (4) $(a, b), (y_i, b), (y_j, d)$; (5) $(y_i, a), (y_j, d)$. If $V_\beta \circ V_\beta \subset V_\alpha$, (1) and (2) imply that (a, d) is in V_α, while (3), (4), and (5) imply that (y_i, a) and (y_j, d) are in V_α. In either case (a, d) is in V'_α. Hence $V'_\beta \circ V'_\beta \subset V'_\alpha$ and the $\{V'_\alpha\}$ define a uniform structure. Although this new uniform structure is not compatible with the topology of $E^\dagger$, its trace on E is useful. For any point x in E, there is a $V_\gamma \varepsilon \mathfrak{V}$ such that $V_\gamma(x)$ does not contain y_1 or y_2. Then $V'_\gamma(x) \subset V_\gamma(x)$. Hence the uniform structures on E defined by the traces of $\mathfrak{V}$ and $\mathfrak{V}'$ on $E \times E$ are both compatible with the topology of E. Since $\mathfrak{V}$ defines the filter of all neighborhoods of $\Delta^\dagger$, there is an entourage V_μ and a neighborhood $W(y_1, y_2)$ in $E^\dagger \times E^\dagger$ which are disjoint. The trace of V_μ on $E \times E$ does not contain the trace of any V'_α. Thus the two uniform structures induced on E are distinct.

THEOREM 3. *If a space has a unique uniform structure it is countably compact.*

Proof. If the space is not countably compact there exists an ordered set $\{x_n\}$ of distinct points without a limit point. Since the space is completely regular there exists an open set $W(x_1)$ such that $\overline{W(x_1)} \cap \bigcup_{j=2}^{\infty} x_j = 0$. For each $n \geqq 2$, there exists an open set $W'(x_n)$ such that $\overline{W'(x_n)} \cap \bigcup_{j=n+1}^{\infty} x_j = 0$. Let $W(x_n)$ be an open set in the intersection of $W'(x_n)$ and the complement of $\bigcup_{j=1}^{n-1} \overline{W(x_j)}$. Then the $\bigcup_{n=1}^{\infty} W(x_{2n})$ and $\bigcup_{n=1}^{\infty} W(x_{2n+1})$ form disjoint open sets which separate two closed sets, $\{x_{2n}\}$ and $\{x_{2n+1}\}$, neither of which is compact. However, according to the condition given by Doss [5], this contradicts the hypothesis.

The converse is not true. Consider a space consisting of two distinct sets of the ordinal numbers of the first and second classes with the usual topology on each set. This space is countably compact but does not have a unique uniform structure.

A space is compact if and only if it has a unique uniform structure relative to which it is complete. The question of determining which topological spaces (weaker than compact spaces) always admit a uniform structure

relative to which they are complete, was raised by André Weil [9], p. 38. Dieudonné [4] showed that normality was not sufficient. In his paper on paracompact spaces [3] he posed the problem of whether or not the universal uniform structure of a paracompact space is the uniform structure of a complete space. The affirmative answer is given here.

THEOREM 4. *A paracompact space E is complete with respect to its universal uniform structure.*

Proof. Suppose the universal uniform structure of E is not complete. Then there exists a Cauchy filter $\mathfrak{F}$ which does not converge; that is, for every x in E, there exists an open set $W(x)$ which does not contain any F_α in $\mathfrak{F}$. Consider the covering $\mathfrak{W}$ consisting of all the open sets $W(x)$ for every x in E. Since E is paracompact, there exists a covering $\mathfrak{U} = \{U_\alpha\}$ such that, for every x in E, the star $(x, \mathfrak{U}) = \bigcup_{x \varepsilon U_\alpha} U_\alpha$ is contained in some set of $\mathfrak{W}$ [6]. Then the open set $U(\Delta) = \bigcup_\alpha (U_\alpha \times U_\alpha)$ is an entourage of the universal uniform structure of E. Since $\mathfrak{F}$ is a Cauchy filter, there is a set F_0 in $\mathfrak{F}$ such that $F_0 \times F_0 \subset U(\Delta)$. Let p be any fixed point in F_0. For every point y in F_0 there is a covering set U_β containing p and y. Since $(p, y) \varepsilon (F_0 \times F_0) \subset U(\Delta)$, the point (p, y) must be an element of some set $U_\beta \times U_\beta$. Thus $F_0 \subset$ star $(p, \mathfrak{U}) \subset W(x)$ for some x, which is absurd.

This gives as immediate corollaries some known [8] relations.

COROLLARY 1. *If a paracompact space is precompact in all its uniform structures it is compact.*

COROLLARY 2. *A paracompact space with a unique uniform structure is compact.*

A converse statement for Theorem 4 poses an interesting problem.

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ON THE ESSENTIAL SPECTRA OF SYMMETRIC OPERATORS IN HILBERT SPACE.*

By Philip Hartman.

The object of this paper is to obtain some results concerning the relationships between the spectra of the self-adjoint extensions of a closed, symmetric operator T on Hilbert space (having equal deficiency indices). These results are known in the case of certain second order, ordinary differential operators T. The proofs in these cases, however, usually depend on functions and operations having no analogues in Hilbert space and, consequently, are not valid in the general case.

1. Point spectra. Let $\mathfrak{H}$ denote a Hilbert space; T a closed, symmetric operator in $\mathfrak{H}$ with domain $\mathfrak{D}(T)$, range $\mathfrak{R}(T)$, adjoint T^* and the deficiency indices (m, m). If l is a complex number, $\mathfrak{M}(l)$ will denote the manifold of elements x of $\mathfrak{H}$ satisfying $(T^* - lI)x = 0$. If $\mathfrak{I}(l) \neq 0$, then the manifolds $\mathfrak{R}(T - lI)$ and $\mathfrak{M}(\bar{l})$ are closed, orthogonal and span $\mathfrak{H}$. In addition, $\mathfrak{D}(T)$, $\mathfrak{M}(l)$, $\mathfrak{M}(\bar{l})$ are in $\mathfrak{D}(T^*)$ and every element x in $\mathfrak{D}(T^*)$ has a unique representation of the form

$$x = x_0 + x(l) + x(\bar{l}), \qquad \mathfrak{I}(l) \neq 0, \tag{1}$$

where x_0, $x(l)$, $x(\bar{l})$ are in $\mathfrak{D}(T)$, $\mathfrak{M}(l)$, $\mathfrak{M}(\bar{l})$, respectively.

For a fixed non-real l, there is a one-to-one correspondence between the self-adjoint extensions A of T and the norm-preserving (linear, continuous) transformations $V = V(l)$ of $\mathfrak{M}(l)$ onto $\mathfrak{M}(\bar{l})$ such that if $A \rightarrow V$, then $\mathfrak{D}(A)$ consists of those elements of the form

$$x = x_0 + x(l) + Vx(l), \qquad \mathfrak{I}(l) \neq 0, \tag{2}$$

where x_0, $x(l)$ are arbitrary elements of $\mathfrak{D}(T)$, $\mathfrak{M}(l)$, respectively (and $Vx(l)$ is in $\mathfrak{M}(\bar{l})$). Cf., e. g., [12], Chap. IX or [9], Chap. VI.

(α) *Let x be an element of $\mathfrak{D}(T^*)$ with the representation* (1) *and satisfying, for some real number λ,*

$$(T^* - \lambda I)x = 0. \tag{3}$$

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Then

$$\| x(l) \| = \| x(\bar{l}) \|. \tag{4}$$

Proof. Without loss of generality it can be supposed that $\lambda = 0$. According to (1) and (3), $-lx = (T^* - lI)x = (T - lI)x_0 - 2i\mathfrak{I}(l)x(\bar{l})$. Hence $\| lx \|^2 = \| (T - lI)x_0 \|^2 + (2\mathfrak{I}(l))^2 \| x(\bar{l}) \|^2$. Similarly,

$$\| lx \|^2 = \| (T - \bar{l}I)x_0 \|^2 + (2\mathfrak{I}(l))^2 \| x(l) \|^2.$$

Since the symmetry of T implies $\| (T - lI)x_0 \| = \| (T - \bar{l}I)x_0 \|$, the result (4) follows.

It is clear from the relationship of $\mathfrak{D}(A)$ to $\mathfrak{M}(l)$ that (α) implies the following assertion:

Corollary. *If x in $\mathfrak{D}(T^*)$ satisfies* (3), *then there exist self-adjoint extensions A of T for which x is in $\mathfrak{D}(A)$, (and, hence, satisfies $(A - \lambda I)x = 0$).*

The next lemma corresponds to a part of Weyl's theorem [14], p. 238, which states that, in his case of differential operators, the deficiency index m can be determined by the consideration of $T^* - \lambda I$ for real λ; cf. (γ) below.

(β) *If the dimension of the manifold of elements x in $\mathfrak{D}(T)$ satisfying $(T - \lambda I)x = 0$ is π, where $0 \leqq \pi \leqq \infty$, then there are at most $m + \pi$ linearly independent solutions of* (3); *so that, if λ is in the point spectrum of a self-adjoint extension A of T, its multiplicity is at least π and at most $m + \pi$.*

Proof. It can be supposed that π and m are finite, for otherwise (β) is trivial. Let $m < \infty$ and $\pi = 0$ (the proof for $\pi > 0$ is similar). Suppose that x satisfies (3) and has the representation (1) for some non-real l. Then $x(l) \neq 0$, for otherwise $x(\bar{l}) = 0$, by (α), and x is in $\mathfrak{D}(T)$, which contradicts the hypothesis, $\pi = 0$. Since there are only m linearly independent elements $x(l)$, it follows that if there were $m + 1$ linearly independent solutions of (3), then there would exist a linear combination of them, say x, for which the corresponding $x(l)$ is 0. This proves (β).

The following assertion is an analogue of results on differential operators; e. g., Weyl [14], pp. 221-227 and Hartman and Wintner, [1]. The proof is adapted from the proof in Stone [12], pp. 489-491, of Weyl's theorem in [14], p. 238.

(γ) *Let A be a self-adjoint extension of T and λ a complex, possibly real, number in the resolvent set of A, then there exist exactly m linearly independent elements x in $\mathfrak{D}(T^*)$ satisfying* (3) (*and no such element is in $\mathfrak{D}(T)$*).

Proof. Only the case of a real λ has to be considered. Let g be an element of the manifold $\mathfrak{M}(l_0)$, where $\mathfrak{I}(l_0) \neq 0$. Let $w = (\lambda - l_0)^{-1}$. Then w is in the resolvent set of the bounded, normal operator $(A - l_0I)^{-1}$ and, therefore, the equation

$$(5) \qquad \{(A - l_0I)^{-1} - wI\}x = g$$

has a unique solution $x = x_g$. This equation can be written as $(A - l_0I)^{-1}x = wx + g$ or $x = (A - l_0I)(wx + g)$. Since $wx + g$ is in $\mathfrak{D}(A)$ (hence, in $\mathfrak{D}(T^*)$) and g is in $\mathfrak{D}(T^*)$, it follows that x is in $\mathfrak{D}(T^*)$. Thus the last equation gives $x = (T^* - l_0I)wx$ or (3), since $w = (\lambda - l_0)^{-1} \neq 0$. Thus, to every g in $\mathfrak{M}(l_0)$, there corresponds an $x = x_g$ of $\mathfrak{D}(T^*)$ satisfying (3). It is clear from (5) that to linearly independent g there correspond linearly independent x_g. Hence (3) has at least m linearly independent solutions x_g.

No element $x \neq 0$ satisfying (3) is in $\mathfrak{D}(T)$, for otherwise $(A - \lambda I)x = (T - \lambda I)x = 0$ contradicts the fact that λ is in the resolvent set of A. Thus (γ) follows from (β).

Remark. The reality of λ was used in the proof above only to assure that $\lambda - l_0 \neq 0$. Thus λ, l_0 can be any pair of numbers in the resolvent set of A with the restriction $\mathfrak{I}(l_0) \neq 0$ and $\lambda \neq l_0$. Writing l in place of λ in w, it follows that $x = x_g(l)$, for a fixed g, depends analytically on l (in the sense that for a fixed element y of $\mathfrak{H}$, the scalar product $(x_g(l), y)$ is a regular analytic function of l). Clearly, there exists a set of m orthonormal elements $x^1(l), x^2(l), \cdots$ spanning $\mathfrak{M}(l)$ and depending continuously on l. Hence, if E_l is the projection on $\mathfrak{M}(l)$, then, for a fixed y, $E_l\, y$ depends continuously on l on the resolvent set of any self-adjoint extension A of T (in particular, on the half-planes $\mathfrak{I}(l) \neq 0$). Cf. [3], [6].

(δ) *Let A_1, A_2 be self-adjoint extensions of T and let l belong to the resolvent set of both A_1, A_2 (for example, let $\mathfrak{I}(l) \neq 0$) and let $D = D(l) = (A_2 - lI)^{-1} - (A_1 - lI)^{-1}$. Then $D = E_lD = DE_{\bar{l}}$.*

In other words, to a complete orthonormal set $x^1, x^2, \cdots$ in $\mathfrak{M}(l)$, there corresponds a set of elements $h_1, h_2, \cdots$ in $\mathfrak{M}(\bar{l})$ such that

$$(6) \qquad Dy = (y, h_1)x^1 + (y, h_2)x^2 + \cdots.$$

In particular, if $m < \infty$, then D is completely continuous.

(δ) and the last remark are generalizations of the result of Weyl [14], p. 251, deduced from the Green kernel representations of $(A_1 - lI)^{-1}$,

$(A_2 - lI)^{-1}$, when $\mathfrak{I}(l) \neq 0$; cf. [2], pp. 314-315 or [6], p. 782, for the corresponding case of a real l.

A modification of the proof of (δ) shows that if $\mathfrak{I}(l) \neq 0$ and V_1, V_2 are the isometric mappings of $\mathfrak{M}(l)$ onto $\mathfrak{M}(\bar{l})$ associated with A_1, A_2, respectively, then $-2\mathfrak{I}(l)Dy = (V_2^{-1} - V_1^{-1})E_l y$.

(ϵ) *If l is real in (δ), then D is bounded, self-adjoint and commutes with E_l. Conversely, if l is real and in the resolvent set of A_1 and D is a bounded, self-adjoint operator which commutes with E_l, then $(A_1 - lI)^{-1} + D$ is the inverse of a self-adjoint operator, say $A_2 - lI$, where A_2 is an extension of T.*

If l is real and $m = 1$, then $D = cE_l$, where c is an arbitrary real constant; for the ordinary differential operator case, see [6], p. 782.

Proof of (δ). Let y be any element of $\mathfrak{H}$ and put $(A_j - lI)^{-1}y = x_j$. Then x_j is in $\mathfrak{D}(A_j - lI)$, hence in $\mathfrak{D}(T^*)$; so that $y = (T^* - lI)x_j$. On letting $j = 1, 2$ and subtracting these relations, it is seen that

$$0 = (T^* - lI)(x_2 - x_1).$$

In other words, $Dy = x_2 - x_1$ is in $\mathfrak{M}(l)$. Hence $D = E_l D$.

In order to prove $D = DE_{\bar{l}}$, it is sufficient to verify that $Dy = 0$ whenever y is orthogonal to $\mathfrak{M}(\bar{l})$. If y is such an element and $\mathfrak{I}(l) \neq 0$, then y is in the range, $\mathfrak{R}(T - lI)$, of $T - lI$, say $y = (T - lI)x$. It follows that $(A_j - lI)^{-1}y = x$ for $j = 1, 2$, and, consequently, $Dy = 0$. If l is real, then y is in the closure of the linear manifold $\mathfrak{R}(T - lI)$, and $Dy = 0$ follows from continuity considerations. This proves (δ).

Proof of (ϵ). If l is real, D is the difference of two bounded, self-adjoint operators; so that the first assertion of (ϵ) is trivial, since $l = \bar{l}$.

Let l and D satisfy the assumptions of the last part of (ϵ). It can be supposed that $l = 0$. Define an operator A_2 by $A_2 = T^*$, where $\mathfrak{D}(A_2)$ consists of those elements x representable in the form $x = (A_1^{-1} + D)y$ for some y in $\mathfrak{H}$. An element x has at most one such representation. In order to see this, note that $x = (A_1^{-1} + D)y$ and $x = (A_1^{-1} + D)z$ imply that $(x - Dy) - (x - Dz) = A_1^{-1}(z - y)$ is in $\mathfrak{D}(A_1)$. Since $D = E_0 D$, $A_1 D(z - y) = T^* D(z - y) = 0$; so that $D(z - y) = 0$ since A_1 is non-singular. Thus $0 = x - x = A_1^{-1}(z - y)$ implies that $z - y = 0$. It is clear that $\mathfrak{D}(A_2)$ contains $\mathfrak{D}(T)$ and is contained in $\mathfrak{D}(T^*)$; so that the definition $A_2 = T^*$ in $\mathfrak{D}(A_2)$ is meaningful. The range of A_2 is $\mathfrak{H}$; thus, if it is

shown that A_2 is symmetric, then (ϵ) will be proved. To this end, let $u = (A_1^{-1} + D)v$, $x = (A_1^{-1} + D)y$ be two elements of $\mathfrak{D}(A_2)$. Then $(A_2u, x) = (v, (A_1^{-1} + D)y) = (u, y) = (u, A_2x)$. This proves ($\epsilon$).

A slight generalization of (ϵ) is given by

(ϵ'). *Let B be a self-adjoint operator such that $Bx = 0$ implies $x = 0$. Let $\mathfrak{M}$ be a (closed) m-dimensional manifold, $0 < m \leqq \infty$, with the property that the equation $Bx = y$ has no solution whenever $y \neq 0$ is in $\mathfrak{M}$. Let $\mathfrak{D}$ be the set of elements x representable in the form $x = Bz$, where z is orthogonal to $\mathfrak{M}$, and let T be the operator, with domain $\mathfrak{D}(T) = \mathfrak{D}$, defined by $Tx = z$ when $x = Bz$. Then T is a closed, symmetric operator with deficiency indices (m, m) and B^{-1} is a self-adjoint extension of T. $\mathfrak{D}(T^*)$ is the set of elements of the form $x + y$, where x is in $\mathfrak{R}(B)$, say $x = Bz$, and y is in $\mathfrak{M}$; finally, $T^*(x + y) = z$.*

The proof is straightforward and will be omitted.

2. A lemma. If A is a self-adjoint operator, let $\Sigma(A)$ denote its spectrum, $\Pi(A)$ its point spectrum, $\Sigma'(A)$ its essential spectrum and $\Gamma(A)$ its continuous spectrum. By the essential spectrum of A is meant the set of finite cluster points of $\Sigma(A)$, including the points in the point spectrum of infinite multiplicity. Thus $\Sigma(A) = \Pi(A) + \Sigma'(A)$, although $\Pi(A)$ and $\Sigma'(A)$ need not be disjoint. By the continuous spectrum of A is meant the set of λ for which the multipilicty of λ in the continuous spectrum is at least 1; cf. [12], p. 267. Thus $\Sigma'(A)$ contains $\Gamma(A)$.

Let X be a closed operator in Hilbert space $\mathfrak{H}$ with a domain $\mathfrak{D}(X)$, dense in $\mathfrak{H}$. It is not supposed that X is symmetric. The object of this section is to point out the relationships between the spectra of the (non-negative) self-adjoint operators X^*X and XX^*. Such relationships are of interest in view of Toeplitz's remarks on the existence of " left " and " right " inverses of X; cf. [16], pp. 138-139, and will have applications below. The results will be based on a theorem of Neumann [10], p. 307, which, in turn, is a generalization of the result of Wintner [17], p. 145 (cf. [15], p. 282), that if X is non-singular (and bounded), then X can be represented as a product of a positive-definite, self-adjoint operator and a unitary operator.

LEMMA. *Let X be a closed operator with a domain $\mathfrak{D}(X)$ which is dense in $\mathfrak{H}$. Let $E(\lambda)$, $F(\lambda)$ be the resolutions of the identity belonging to X^*X, XX^*, respectively (where $E(\lambda)$, $F(\lambda)$ are continuous from the right). Then there exists a unique bounded operator W with the properties that*

$$W^*W = I - E(0) \text{ and } WW^* = I - F(0) \tag{7}$$

and that if $0 \leqq \lambda \leqq \mu$, *then*

$$(8)\quad F(\mu) - F(\lambda) = W\{E(\mu) - E(\lambda)\}W^* \textit{ and } E(\mu) - E(\lambda) = W^*\{F(\mu) - F(\lambda)\}W.$$

These relations imply that if $y = Wx$ (or if $x = W^*y$) and $0 \leqq \lambda < \mu$, then $\| \{E(\mu) - E(\lambda)\}x \| = \| \{F(\mu) - F(\lambda)\}y \|$. Hence, every $\lambda \neq 0$ has the same multiplicity in the point spectra $\Pi(X^*X)$ and $\Pi(XX^*)$ and every λ has the same multiplicity in the continuous spectra $\Gamma(X^*X)$ and $\Gamma(XX^*)$. The exceptional standing of $\lambda = 0$ arises, of course, from the fact that although $E(-0) = F(-0) = 0$, the projections $E(0)$, $F(0)$ need not be 0 and, in fact, their ranges can be manifolds of different dimensionality. Thus $\lambda = 0$ can be in one of the sets $\Pi(X^*X)$, $\Pi(XX^*)$ without being in the other or, more generally, the multiplicities of $\lambda = 0$ in the sets $\Pi(X^*X)$, $\Pi(XX^*)$ can be different. Except for this possibility, the spectra (counting multiplicities) of X^*X and XX^* are identical.

It is clear from the Lemma (without any appeal to the Hellinger theory [8] or its extensions) that X^*X and XX^* are unitarily equivalent if and only if the multiplicities of $\lambda = 0$ in $\Pi(X^*X)$ and $\Pi(XX^*)$ are equal. For if these multiplicities are equal, the definition of W (which is 0) on the range of $E(0)$ can be altered so that W gives an isometric mapping of the range of $E(0)$ onto the range of $F(0)$. The altered W, say U, will be unitary and will satisfy $XX^* = U^*X^*XU$.

In this case (when the multiplicities of $\lambda = 0$ in $\Pi(X^*X)$ and $\Pi(XX^*)$ are equal), the theorem of Wintner mentioned above has the extension that X is the product of a non-negative self-adjoint operator and of a unitary operator, in fact, $X = (XX^*)^{\frac{1}{2}}U = U^*(X^*X)^{\frac{1}{2}}$.

The Lemma and its consequences were suggested to me by a discussion with Professor Wintner of the Appendix of [11], pp. 75-78.

Proof of the Lemma. According to [10], p. 307, there exists a (unique) bounded operator W with the properties that it maps $\mathfrak{R}((X^*X)^{\frac{1}{2}})$ isometrically onto $\mathfrak{R}((XX^*)^{\frac{1}{2}})$ and satisfies (7) and $X = W(X^*X)^{\frac{1}{2}} = (XX^*)^{\frac{1}{2}}W^*$. Furthermore, for every x in $\mathfrak{H}$, the element Wx is in the closed linear manifold spanned by $\mathfrak{R}((XX^*)^{\frac{1}{2}})$ and, hence, is orthogonal to the range of $F(0)$ (that is, to the elements satisfying $(XX^*)^{\frac{1}{2}}y = 0$ or equivalently $X^*y = 0$). Thus $F(0)W = 0$. From (7), it is clear that $W^*F(0) = 0$. These two relations, their analogues $E(0)W^* = WE(0) = 0$ and the relations (7) make it clear that if $G(\lambda)$ is defined to be 0 or $F(0) + WE(\lambda)W^*$ according as $\lambda < 0$ or $\lambda \geqq 0$, then $G(\lambda)$ is a resolution of the identity. It follows from

$XX^* = WX^*XW^*$ that $G(\lambda) \equiv F(\lambda)$, that is, that $F(\lambda)$ is 0 or $F(0) + WE(\lambda)W^*$ according as $\lambda < 0$ or $\lambda \geqq 0$. This implies (8) and completes the proof of the Lemma.

Remark. The theorem of Neumann, referred to above, has also the following consequences: For a given y, the existence of a solution x or z for one of the equations

$$X^*x = y \quad \text{and} \quad (X^*X)^{\frac{1}{2}}z = y$$

implies the existence of a solution z or x for the other equation. This follows from $X^* = (X^*X)^{\frac{1}{2}}W^*$. For if x is a solution for the first equation then $z = W^*x$ is a solution for the second. If z is the solution of least norm of the second equation, so that $(I - E(0))z = z$, then $x = Wz$ satisfies $W^*x = W^*Wz = z$, by (7), and is a solution of the first equation. In other words, $\mathfrak{R}((X^*X)^{\frac{1}{2}}) = \mathfrak{R}(X^*)$ (this contrasts with the relation $\mathfrak{D}((X^*X)^{\frac{1}{2}}) = \mathfrak{D}(X)$; [10], p. 304). Thus $X^*x = y$ fails to have a solution for some y if and only if $\lambda = 0$ is in the spectrum $\Sigma((X^*X)^{\frac{1}{2}})$, that is, in $\Sigma(X^*X)$.

3. Essential spectra. As in Section **1**, T will denote a closed, symmetric operator with deficiency indices (m, m). Let the set of (real) λ for which there is an $x \neq 0$ in $\mathfrak{H}$ satisfying

$$(T - \lambda I)x = 0, \tag{9}$$

be called $\Pi(T)$. Correspondingly, let the set of *real* λ for which there is an $x \neq 0$ satisfying (3) be called $\Pi_0(T^*)$. Let $\Sigma_0(T^*)$ denote the set of *real* λ to which there corresponds at least one element y in H with the property that

$$(T^* - \lambda I)x = y \tag{10}$$

has no solution x.

It can be remarked that $\lambda = \mu$ is in $\Pi(T)$ if and only if $\lambda = 0$ is in $\Pi((T^* - \mu I)(T - \mu I))$, the point spectrum of the self-adjoint operator $(T^* - \mu I)(T - \mu I)$. Similarly, $\lambda = \mu$ is in $\Pi_0(T^*)$ if and only if $\lambda = 0$ is in $\Pi((T - \mu I)(T^* - \mu I))$. In view of the Remark at the end of the last section, $\lambda = \mu$ is in $\Sigma_0(T^*)$ if and only if $\lambda = 0$ is in $\Sigma((T^* - \mu I)(T - \mu I))$.

Let Σ_1' and Σ_2' denote the sets of μ-values for which $\lambda = 0$ is in $\Sigma'((T^* - \mu I)(T - \mu I))$ and $\Sigma'((T - \mu I)(T^* - \mu I))$, respectively. Since $\Pi(T)$ is contained in $\Pi_0(T^*)$, it follows from the Lemma of the last section that Σ_1' is contained in Σ_2'. Furthermore, if $m < \infty$, then $\Sigma_1' = \Sigma_2'$ by virtue of (β), since a necessary condition for a $\lambda = \mu$ in Σ_2' to fail to belong to Σ_1'

is that $\lambda = \mu$ be in $\Pi(T)$ with a finite, and in $\Pi_0(T^*)$ with an infinite multiplicity.

In what follows, A_1, A_2 and A represent self-adjoint extensions of A.

(i) *If $0 < m < \infty$ and if the interval $\mu \leqq \lambda \leqq \nu$ contains $m + 1$ points of the spectrum of A_1 (where points of the point spectrum are counted with their multiplicities), then the same interval contains at least one point of the spectrum of A_2. In particular, $\Sigma'(A_1) = \Sigma'(A_2)$.*

The first part of this assertion is clear from (δ). The last part is a consequence of the first part; in this assertion, "finite cluster points" in the definition of the essential spectrum $\Sigma'(A)$ can be replaced by "finite and infinite cluster points." (This last remark does not follow if one does not use the full force of (δ), but only its consequence that D is completely continuous.) Thus if A_1 is bounded (or unbounded) from above, so is A_2.

The last part of the italicized assertion above is a generalization of a result of Weyl [14], p. 251, for differential operators. The generalization (i) to arbitrary symmetric operators of the type here considered is due to E. Heinz [7]. Other separation theorems for the case $m = 1$ are given in [4], [6].

(ii) *A point $\lambda = \mu$ is in at least one $\Pi(A)$ if and only if it is in $\Pi_0(T^*)$, that is, if and only if $\lambda = 0$ is in $\Pi((T - \mu I)(T^* - \mu I))$. A point $\lambda = \mu$ is in every $\Pi(A)$ if and only if it is in $\Pi(T)$, that is, if and only if $\lambda = 0$ is in $\Pi((T^* - \mu I)(T - \mu I))$.*

The situation with respect to essential spectra is somewhat more complicated, when $m = \infty$:

(iii) *If a point $\lambda = \mu$ is in at least one $\Sigma'(A)$, then it is in Σ_2', that is, $\lambda = 0$ is in $\Sigma'((T - \mu I)(T^* - \mu I))$. If a point $\lambda = \mu$ is in Σ_1', that is, if $\lambda = 0$ is in $\Sigma'((T^* - \mu I)(T - \mu I))$, then it is in every $\Sigma'(A)$.*

When $0 < m < \infty$, then $\Sigma'(A)$ is independent of A and is identical with the set $\Sigma_1' = \Sigma_2'$.

(iv) *Every point $\lambda = \mu$ is in at least one $\Sigma(A)$ and, for every μ, the point $\lambda = 0$ is in $\Sigma((T - \mu I)(T^* - \mu I))$. If a point $\lambda = \mu$ is in $\Sigma_0(T^*)$, that is, if $\lambda = 0$ is in $\Sigma((T^* - \mu I)(T - \mu I))$, then $\lambda = \mu$ is in every $\Sigma(A)$; conversely, when $0 < m < \infty$, if a point $\lambda = \mu$ is in $\Sigma(A)$ for every A, then it is in $\Sigma_0(T^*)$.*

These assertions are known for certain ordinary differential operators, where $m = 1$; Hartman and Wintner [5].

Proof of (ii). Since T^* is an extension of A, it is clear that $\Pi_0(T^*)$ contains $\Pi(A)$. Conversely, if $\lambda = \mu$ is in $\Pi_0(T^*)$, then it is in some $\Pi(A)$ by the Corollary of (α). This proves the first assertion of (ii).

Clearly, if $\lambda = \mu$ is in $\Pi(T)$, then it is in every $\Pi(A)$. In order to see that if $\lambda = \mu$ is not in $\Pi(T)$, then there is an A for which $\lambda = \mu$ is not in $\Pi(A)$, let $x^1, x^2, \cdots$ be a complete set of linearly independent elements in $\mathfrak{M}(\mu)$. For some non-real l, let $x^j = x_0^j + x^j(l) + x^j(\bar{l})$ be the decomposition (1) of x^j. Suppose $x^1, x^2, \cdots$ have been selected so that $x^1(l)$, $x^2(l), \cdots$ form an orthonormal sequence. Then, by (α), $x^1(\bar{l}), x^2(\bar{l}), \cdots$ is also an orthonormal sequence. Consider an isometric mapping V_1 of $\mathfrak{M}(l)$ onto $\mathfrak{M}(\bar{l})$, for which $V_1 x^j(l) = x^j(\bar{l})$. Then any isometric mapping V of $\mathfrak{M}(l)$ onto $\mathfrak{M}(\bar{l})$ can be represented as $V = UV_1$, where U is a unitary mapping of $\mathfrak{M}(\bar{l})$ onto $\mathfrak{M}(\bar{l})$. Let U be chosen so that $\lambda = 1$ is not in its point spectrum. Then it readily follows that no element $x \neq 0$ in $\mathfrak{M}(\mu)$ is in $\mathfrak{D}(A)$, where A is the self-adjoint extension of T corresponding to V. Thus $\lambda = \mu$ is not in $\Pi(A)$.

Proof of (iii). The proof will depend on the following criterion of Weyl, [13]: For any self-adjoint operator B, $\lambda = 0$ is in $\Sigma'(B)$ if and only if there exists a sequence $x_1, x_2, \cdots$ in $\mathfrak{D}(B)$ such that $\| x_n \| = 1$, $x_n \to 0$ (weakly) and $Bx_n \to 0$ (strongly), as $n \to \infty$.

It can be supposed that $\mu = 0$. The second assertion of (iii) will be proved first. Suppose that $\lambda = 0$ is in Σ_1', that is, in $\Sigma'(T^*T)$. Then there is a sequence $x_1, x_2, \cdots$ with the properties described above, where $B = T^*T$. Since $\| x_n \| = 1$ and $T^*Tx_n \to 0$ (strongly), as $n \to \infty$, it follows that $(T^*Tx_n, x_n) \to 0$ as $n \to \infty$; that is, $Tx_n \to 0$ (strongly) as $n \to \infty$. Since every A is an extension of T, it follows that $Ax_n \to 0$ (strongly), as $n \to \infty$. Thus $\lambda = 0$ is in $\Sigma'(A)$ for every A.

The proof of the first assertion of (iii) is proved similarly. Let $\mu = 0$ and let $x_1, x_2, \cdots$ have the properties described in Weyl's criterion, where $B = A$ for some A. Then x_n, which is in $\mathfrak{D}(A)$, is in $\mathfrak{D}(T^*)$ and satisfies $T^*x_n = Ax_n$. Hence $T^*x_n \to 0$ (strongly), which implies that $(TT^*)^{\frac{1}{2}}x_n \to 0$ (strongly); cf. [10], p. 304. Thus $\lambda = 0$ is in $\Sigma'((TT^*)^{\frac{1}{2}})$ and, hence, in $\Sigma'(TT^*)$. Consequently, $\lambda = 0$ is in Σ_2', as was to be proved.

Proof of (iv). If $\lambda = \mu$ is not in $\Sigma(A)$, for some A, then $\lambda = \mu$ is in $\Pi_0(T^*)$, by (γ). Hence $\lambda = \mu$ is in $\Pi(A_1)$ for some A_1, by (α). This proves the first part of the first assertion in (iv).

In order to prove the second part, note that if $\lambda = 0$ is not in $\Pi((T - \mu I)(T^* - \mu I))$, then $\lambda = \mu$ is in $\Sigma(A)$ for every A, by (γ), and hence, is in $\Sigma'(A)$, by (ii). Consequently, (iii) implies that $\lambda = 0$ is in $\Sigma'((T - \mu I)(T^* - \mu I))$; in others words, for every μ, $\lambda = 0$ is in

$$\Pi((T - \mu I)(T^* - \mu I)) + \Sigma'((T - \mu I)(T^* - \mu I)) = \Sigma((T - \mu I)(T^* - \mu I)).$$

There remains to prove the second assertion of (iv), that concerning $\Sigma_0(T^*)$. The last parts of (ii) and (iii) show that if $\lambda = 0$ is in $\Sigma((T^* - \mu I)(T - \mu I))$, then $\lambda = \mu$ is in $\Sigma(A)$ for every A. When $m < \infty$, the converse is true, since $m < \infty$ implies $\Sigma_1' = \Sigma_2'$. Hence (iv) follows from the Remark at the end of the last section (if X is identified with $T - \mu I$).

4. Continuous spectra. For a self-adjoint operator A and a real number λ, let $p(\lambda) = p(\lambda, A)$ and $c(\lambda) = c(\lambda, A)$ denote the multiplicities of λ in the point and continuous spectra of A, respectively; cf., e. g., [12], p. 267. (The continuous spectrum $\Gamma(A)$ is the set of λ for which $c(\lambda) \geqq 1$.)

Let T be a closed symmetric operator with deficiency indices (m, m), where $0 < m \leqq \infty$. For a real λ, let

$$(11) \qquad \pi(\lambda) = \min_A p(\lambda, A); \qquad\qquad (12) \qquad \gamma(\lambda) = \min_A c(\lambda, A),$$

where the minimum is taken over all self-adjoint extensions A of T $(0 \leqq \pi(\lambda) \leqq \infty,\ 0 \leqq \gamma(\lambda) \leqq \infty)$.

(I) *If A is a self-adjoint extension of T, then*

$$(13) \quad \pi(\lambda) \leqq p(\lambda, A) \leqq \pi(\lambda) + m; \qquad (14) \quad \gamma(\lambda) \leqq c(\lambda, A) \leqq \gamma(\lambda) + m.$$

Proof. The relation (13) is a consequence of (β) and its proof; in fact, $\pi(\lambda)$ is the number π occurring in that assertion.

In order to prove (14), let A_1 be a self-adjoint extension of T such that

$$(15) \qquad c(\lambda, A_1) = \gamma(\lambda).$$

It will be shown that, for every A,

$$(16) \qquad c(\lambda, A_1) \geqq c(\lambda; A) - m.$$

It can be supposed that $m < \infty$ and that $c(\lambda, A) \geqq m$; for otherwise (16), which is equivalent to (14), is trivial.

Let $E(\lambda)$, $E_1(\lambda)$ denote the spectral resolutions of A, A_1, respectively. Let $y^1, y^2, \cdots$ denote (the possibly empty) set of eigenfunctions of A and let $\mathfrak{N}$ denote the closed manifold spanned by them. Let $x_1, x_2, \cdots$ be a sequence of elements with the properties that the sets $E(\lambda)x_k$, where $-\infty < \lambda < \infty$ and $k = 1, 2, \cdots$ span the closed linear manifold orthogonal to

$\mathfrak{N}$, that $(E(\lambda)x_k, x_j) = 0$ for $-\infty < \lambda < \infty$ and $j \neq k$, that $\sigma_k(\lambda) = \| E(\lambda)x_k \|^2$ is absolutely continuous with respect to $\sigma_{k-1}(\lambda)$. Then $c(\lambda, A)$ is the number of (continuous) functions $\sigma_1, \sigma_2, \cdots$ which are not constant on any open interval containing λ.

Let $\mathfrak{M}(x)$ denote the closed linear manifold spanned by the elements $E(\lambda)x$, where $-\infty < \lambda < \infty$. If $y_1, y_2, \cdots, y_m$ are m given elements orthogonal to $\mathfrak{N}$, then it can be supposed that the manifold spanned by $\mathfrak{M}(x_1), \cdots, \mathfrak{M}(x_m)$ contains $\mathfrak{M}(y_1), \cdots, \mathfrak{M}(y_m)$; cf. [12], pp. 247-262.

Let $\mathfrak{d}(l) \neq 0$ and let $x^1, x^2, \cdots, x^m$ be m linearly independent elements in $\mathfrak{M}(\bar{l})$. Let the elements $y_1, y_2, \cdots, y_m$ of the last paragraph be chosen to be the respective components of $x^1, \cdots, x^m$ in the manifold orthogonal to $\mathfrak{N}$.

If $k > m$, then $E(\lambda)x_k$ for every λ is orthogonal to $x^1, \cdots, x^m$; hence, the same is true of $(A - lI)^{-n}x_k$ for $n = 0, 1, \cdots$. By (δ), it follows that $(A_1 - lI)^{-n}x_k = (A - lI)^{-n}x_k$ for $n = 0, 1, \cdots$ and so, $E_1(\lambda)x_k = E(\lambda)x_k$ for $-\infty < \lambda < \infty$ and $k > m$. It follows from a standard procedure for determining $c(\lambda, A)$, that (16) and, therefore, (14) holds; cf. [12], pp. 247-262.

Remark. Let $\pi_0(\lambda) = \max p(\lambda, A)$ and $\gamma_0(\lambda) = \max c(\lambda, A)$, where the maxima are taken over all self-adjoint extensions A of T. Then $\pi(\lambda)$ and $\pi_0(\lambda)$ are, respectively, the dimensions of the manifolds determined by (7) and (3); that is, the multiplicities of λ in $\Pi(T)$ and $\Pi(T^*)$. For any integer k satisfying $\pi(\lambda) \leqq k \leqq \pi_0(\lambda)$, there is an $A = A(k, \lambda)$ such that $p(\lambda, A) = k$; cf. (α) and (β).

There naturally arises the question of the determination of $\gamma(\lambda)$ and $\gamma_0(\lambda)$, say in terms of T and T^*, and whether there exists an $A = A(k)$ such that $c(\lambda, A) = k$ when $\gamma(\lambda) \leqq k \leqq \gamma_0(\lambda)$. The answers to these questions would, in turn, answer the question raised by Weyl [14], pp. 251-252, in connection with his differential operators, as to whether or not the continuous spectrum $\Gamma(A)$ is independent of the boundary condition determining the self-adjoint extension A. They would also answer the somewhat more general question raised by Wintner as to the dependency of the continuous spectra $\Gamma(A + \tau E)$ on τ, where A is an arbitrary self-adjoint operator, E is a 1-dimensional projection and τ is a real number.

In this regard, (ii)-(vi) suggest that if $m < \infty$, then $\lambda = \mu$ is in every $\Gamma(A)$ if and only if $\lambda = 0$ is in $\Gamma((T^* - \mu I)(T - \mu T))$ and that $\gamma(\lambda) = \gamma_0(\lambda)$.

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ON THE INFINITESIMAL GEOMETRY OF CURVES.*

By Aurel Wintner.

1. Let Γ be an arc of class C'' in the X-space, where $X = (x, y, z)$, that is, let Γ be a rectifiable Jordan arc having the property that the second derivative X'' of $X = X(s)$, where s is the arc length $(|X'| = 1)$, exists and is continuous. In particular, $\kappa = |X''|$ defines a continuous non-negative function $\kappa(s)$, the curvature on Γ. Suppose that $X'' \neq 0$ on Γ; so that

$$\kappa = |X''| > 0, \tag{1}$$

and so Frenet's mutually perpendicular unit vectors

$$U_1 = X', \qquad U_2 = \kappa^{-1}X'', \qquad U_3 = [U_1, U_2] \tag{2}$$

(where the bracket refers to vector multiplication, hence $\det(U_1, U_2, U_3)$ is $+1$) exist and represent *continuous* functions $U_k(s)$ of s (in addition, $U_1(s)$ is of class C'). But Frenet's differential equations for (2) do not exist under the present assumption, since no torsion can be defined for an arbitrary Γ of class C'' satisfying (1).

The classical theory therefore assumes that Γ is of class C''' (i. e., that there exists a continuous third derivative $X'''(s)$). Then the torsion can be defined as $\det(X', X'', X''')/|X''|^2$ and is a continuous function, whereas the curvature $|X''| > 0$ (instead of being just continuous as above) is a function of class C'. But the restriction of the theory of torsion to curves Γ of class C''' has disagreeable consequences in the theory of surfaces, for instance, if Γ is an asymptotic line or a geodesic, since, as pointed out in [2], p. 772 and [4], p. 608, respectively, the surface must then be required to be unnaturally smooth before the classical facts concerning these two types of curves on a surface can be formulated at all. In addition, one would surely like to (but, on the classical basis, cannot) say that a plane curve Γ has a torsion $(\equiv 0)$ even if Γ is of class C'' only.

For these reasons, the following definition was introduced in [2], p. 771 (and will always be used in what follows): A curve Γ, of class C'' and of

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non-vanishing curvature (1), is said to have a continuous torsion $\tau = \tau(s)$ if the vector functions (2) of s are such that

$$\tau = \lim_{\Delta s \to 0} \det(U_1, U_2, \Delta U_2/\Delta s) \tag{3}$$

where $U_k = U_k(s)$ and $\Delta U_2 = U_2(s + \Delta s) - U_2(s)$, exists and is a continuous function of s. The results of [2], pp. 773-774, imply that this will be the case if and only if

$$\lim_{\Delta s \to 0} \Delta U_2/\Delta s \tag{3 bis}$$

exists and is continuous, that is, if and only if $U_3 = [U_1, U_2]$ is a function of class C'. This means that (3) can be simplified to

$$\tau = \det(U_1, U_2, U_2'). \tag{4}$$

It may be mentioned that the above definition of *torsion* is the more natural from the geometrical point of view as it is the precise analogue of the definition of the Gaussian curvature, at non-parabolic points, of a surface of class C'' in terms of oriented spherical images. In fact, if a curve of class C'' has a non-vanishing curvature (and the latter itself is defined in terms of spherical images, those belonging to the tangent vector U_1), then (2) defines three continuous unit vectors. If the third of them, the binormal, is a function of class C' and s is not a stationary point, so that $U_3'(s) \neq 0$, then, near that point, $N = U_3(s)$ is a Jordan arc of class C' on the unit sphere $|N| = 1$, and the ratio of the oriented arc lengths of the portions $(s, s + \Delta s)$ of this curve and of the given curve, being equal to

$$\pm \int_s^{s+\Delta s} |U_3'(t)|\, dt / |\Delta s|,$$

tends to $\tau(s)$ as $\Delta s \to 0$. But nothing in this geometrical definition requires the usual assumption of a third derivative for $X(s)$.

2. We did not emphasize in [2] the following explicit criterion: *A curve of class C'' and of non-vanishing curvature possesses a continuous torsion if and only if all three unit vectors $U_k = U_k(s)$ are functions of class C'.* In fact, U_1 is of class C' whenever Γ is of class C'', and so the C'-character of U_3 follows from that of U_2, since $U_3 = [U_1, U_2]$, by (2).

Note that, in contrast to the situation in the C'''-theory, the curvature (1) and the torsion (4) are now of equal smoothness, namely, just continuous. As shown in [2], p. 772, the two continuous scalar functions $\kappa > 0$, τ and

the three continuously differentiable vector functions U_k satisfy Frenet's equations

$$(5)\qquad U_1' = \kappa U_2, \qquad U_2' = -\kappa U_1 + \tau U_3, \qquad U_3' = -\tau U_2$$

and, conversely, the assignment of any pair of continuous functions $\kappa > 0$, τ of s determines, via (5), a unique curve Γ: $X(s)$ of class C'' satisfying (1), (2) and (4) (the uniqueness of this $\Gamma = \Gamma(\kappa, \tau)$ is meant modulo the group of movements in the Euclidean X-space).

It was shown in [2] and [4], respectively, that this definition of a continuous torsion eliminates the above-mentioned objections to the classical theory of asymptotic curves and of geodesics. Furthermore, since $U_3(s) = \text{const.}$ in case of a plane curve of class C'' satisfying (2), it is clear from (4) that every such curve has the torsion $\tau(s) \equiv 0$ (the converse, too, is true; it follows from the existence and uniqueness theorems, mentioned after (5) above).

In terms of the approximate expansion of Γ near a point s of Γ, the replacement of the classical C'''-assumption by the more inclusive C''-class of continuous torsion can be characterized as follows: With reference to a fixed s, let the coordinate system $X = (x, y, z)$ be so chosen that $X(s) = (0, 0, 0)$ and that the coordinate axes x, y, z are those determined by the respective unit vectors $U_1(s)$, $U_2(s)$, $U_3(s)$. Then, under the assumption that Γ is of class C''' and of non-vanishing curvature, the approximate expressions of the three components of $X(s+h) - X(s) = X(s+h)$ are known to be

$$(5\,\text{bis})\qquad h - \kappa^2 h^3/6, \qquad \tfrac{1}{2}\kappa h^2 + \kappa' h^3/6, \qquad \tau\kappa h^3/6,$$

respectively, with error terms *all three* of which are $o(|h|^3)$ as $h \to 0$ (the values of the functions κ, τ and of the continuous derivative κ' refer here to the point s). If the more inclusive case of Section 1 is considered, that is *if Γ is a curve of class C'' having a non-vanishing curvature and a continuous torsion, then, while the approximation* (5 bis), *with three $o(|h|^3)$-terms, need not hold, it holds in the curtailed form*

$$(5^*)\qquad h - \kappa^2 h^3/6 + o(|h|^3), \qquad \tfrac{1}{2}\kappa h^2 + o(h^2), \qquad \tau\kappa h^3/6 + o(|h|^3).$$

These asymptotic formulae, which can readily be deduced from (5), (2) and (1), suffice for the characterization of the values of $\kappa = \kappa(s) \neq 0$ and $\tau = \tau(s)$.

3. The precise nature of the classical C'''-assumption will now be analyzed explicitly, by proving the following criterion: *In order that a curve Γ of class C'' satisfying* (1) *be of class C''', the following pair of independent*

assumptions is necessary and sufficient: (i) Γ *has a continuous torsion* $\tau = \tau(s)$ *and* (ii) *the curvature* $\kappa = \kappa(s)$ *of* Γ *is of class* C'.

The necessity of both (i) and (ii) is the statement of the classical theory. The insufficiency of (i) alone is shown by the example of plane curves Γ which are of class C'' but not of class C'''. The insufficiency of (ii) alone can be concluded from the following example: On the closed interval $0 \leqq s \leqq 1$, let $\kappa(s)$ be a positive function possessing a continuous first derivative (e.g., $\kappa(s) \equiv 1$), and let $\tau(s)$ be a function which is bounded for $0 \leqq s \leqq 1$, continuous for $0 < s \leqq 1$ but discontinuous at $s = 0$ (such as $\tau(s) = \sin 1/s$, where $s \neq 0$, and, for instance, $\tau(0) = 0$). With reference to this pair, the linear differential equations (5) define on the interval $0 < s \leqq 1$ a curve $X = X(s)$ of class C'' satisfying (1), (2) and (4), where $U_1 = X'$. Moreover, if (5_k) denotes the k-th of the three equations (5), application of two quadratures to (5_1) on the interval $\epsilon \leqq s \leqq 1$, when followed by the limit process $\epsilon \rightarrow 0$, shows that the limit $X(+0)$ exists and, if it is declared to be $X(0)$, then $X(s)$ is of class C'' on the closed interval $0 \leqq s \leqq 1$; and that (5_1), where $U_1 = X'$ and $U_2 = \kappa^{-1}X''$, holds at $s = 0$ also. In particular, U_1 is of class C', and U_2 is continuous, for $0 \leqq s \leqq 1$. But this curve Γ: $X(s)$ cannot be of class C''' on $0 \leqq s \leqq 1$. For if it were, it ought to possess a continuous torsion at $s = 0$ also. In particular, (5_3) would hold at $s = 0$ with a continuous U_3'. Since (5_3) implies that $\tau = -U_2 \cdot U_3'$, this leads to the existence of the limit $\tau(+0)$, which contradicts the assumption.

This proves that (ii) alone is insufficient, and so it only remains to be shown that (ii) and (i) together are sufficient, in order that Γ be of class C'''. But this can be seen as follows: As pointed out above, (i) means that all three functions U_k of s are of class C'. In particular, U_2 is of class C'. It follows therefore from (5_1) and (ii) that U_1' is of class C'. Since $U_1' = X''$, this proves that X is of class C'''.

4. Let Γ: $X = X(s)$ be an arc of class C''' satisfying both (1) and

$$\tau \neq 0. \tag{6}$$

Then a classical result (cf. [1], pp. 28-40) states that, corresponding to every point s of Γ, there exist an unique "osculating sphere," that is, one and only one sphere having the property that the distance between the point $X(s + \Delta s)$ of Γ and that sphere is $o(|\Delta s|^3)$ as $\Delta s \rightarrow 0$. The vector, say Y, representing the center of this sphere is known to be

$$Y = X + \kappa^{-1}U_2 - (\kappa^2\tau)^{-1}\kappa' U_3 \tag{7}$$

(cf. ibid.). Clearly, $Y = Y(s)$ is a continuous function. Let Γ_0 be the set of points described by $Y = Y(s)$ when s ranges over Γ. In what follows, it will be tacitly assumed that the locus $\Gamma_0 = \Gamma_0(\Gamma)$ is referred to a sufficiently short Γ. Although $Y(s)$ is a continuous function, Γ_0: $Y = Y(s)$ need not be a Jordan arc (for instance, Γ_0 is a single point when Γ is a circular arc).

In order to obtain an explicit condition preventing this contingency, suppose that Γ is of class C^4. Then $\kappa(s)$ will have a continuous second, and $\tau(s)$ a continuous first, and so the function (7) a continuous first, derivative. In order to obtain the latter, note that the derivative of the first term of (7) is $X' = U_1$, and the derivatives of the factors U_2, U_3 occurring in the second and third terms of (7) are given by (5_2) and (5_3). After trivial reductions, this supplies for the derivative $Y' = Y'(s)$ of $Y = Y(s)$ the representation

(8) $$Y' = \sigma U_3,$$

if the scalar σ denotes the difference

(9) $$\sigma = \tau\kappa^{-1} - (\tau^{-1}\kappa^{-2}\kappa')'$$

which, $X(s)$ being a function of class C^4 satisfying (1) and (6), is a continuous function $\sigma = \sigma(s)$.

If Γ: $X = X(s)$ satisfies the additional condition

(10) $$\sigma \neq 0$$

at a point s (hence near that point), then (8) shows that $Y' \neq 0$, which assures that Γ_0: $Y = Y(s)$ is a Jordan arc and, what is more, that Γ_0 is an arc possessing a continuous tangent (in fact, Y' is a continuous function of s).

A Jordan arc Γ: $X = X(s)$ will be called *a curve of class C^n free of spherical points* [1] if the function $X(s)$ has a continuous n-th derivative and satisfies (10) for every s. This implies that $n \geqq 4$ and that neither the curvature κ nor the torsion τ vanishes on Γ, since otherwise the function (9) cannot even be defined.

[1] In order to arrive at an interpretation of assumption (10), or rather of its violation, $\sigma(s) = 0$, at a *single* point s, it is sufficient to observe that $\sigma(s) \equiv 0$, the *identical* vanishing of the function $\sigma(s)$, is characteristic of those curves Γ of class C^4 satisfying (1) and (6) which are situated on a fixed sphere (cf. [1], p. 41). A point s of Γ can therefore be called a "spherical" or a "non-spherical" point according as it violates or satisfies condition (10). The situation becomes clear if, corresponding to this terminology, a point s of Γ is called a "planar" or a "non-planar" point according as it violates or satisfies condition (6) (where only (1) and the C'''-character of Γ need be assumed).

5. If the torsion of a curve is defined in the classical manner (that is to say so as to assume the C'''-character of the curve), then, in order to be able to speak of the torsion of the curve Γ_0, the curve Γ is restricted to be of class C^6, since 3 degrees of differentiability appear to be sacrificed in the passage from $X(s)$ to $Y(s)$; cf. (7). But it turns out that, on the one hand, this appearance is misleading, since $n=6$ can be reduced to $n=4$, and that, on the other hand, definition (4) of the torsion allows a reduction by one more degree of differentiability. In other words, the restriction to the class C^6, which is a tacit assumption of Monge's theory (cf. [1], pp. 42-43), can actually be reduced to the class C^4 which, by the end of Section 4, represents the optimum in this context (at least if $\sigma=\sigma(s)$, the "measure of non-spherical character," is defined by the explicit formula (9)). In fact, it will be proved that *if a curve Γ of class C^4 is free of spherical points, then Γ_0, the locus of the centers of the osculating spheres of Γ, is a curve of class C'' possessing a non-vanishing continuous curvature and a non-vanishing continuous torsion.* If the latter are denoted by κ_0 and τ_0, respectively, their explicit representations in terms of curvature and the torsion of Γ itself prove to be, as in the classical case (cf. [1], pp. 42-43),

(11) $\qquad \kappa_0=|\tau/\sigma|, \qquad\qquad$ (12) $\qquad \tau_0=\kappa/\sigma.$

6. The first assertion of the italicized theorem, namely, the C''-character of Γ_0, can be proved as follows: In view of (10), the quadrature assigned by

$$ds_0=\sigma ds, \tag{13}$$

where $\sigma=\sigma(s)$ is continuous, establishes between the arc length s on Γ and the parameter s_0 a one-to-one correspondence in such a way that the function $s_0=s_0(s)$ and its inverse $s=s(s_0)$ have continuous first derivatives $(\neq 0)$ with respect to s and s_0, respectively. It is also seen from (13) and (10) that, generally,

$$F'=\sigma F^{\wedge}, \text{ where } \sigma\neq 0 \text{ and } F^{\wedge}=dF/ds_0, \qquad F'=dF/ds. \tag{14}$$

Hence, (8) can be written in the form

$$Y^{\wedge}=U_3. \tag{15}$$

In view of $|U_3|=1$, this implies that $|Y^{\wedge}|=1$, which means that s_0 is the arc length on Γ_0. It also follows from (15) that, in order to prove that $Y=Y(s_0)$ is of class C'', it is sufficient to ascertain that U_3 is of class C' as a function of s_0. But this is obvious, since $U_3=U_3(s)$ and $s=s(s_0)$

are functions of class C' of s and s_0, respectively. This proves that Γ_0 is an arc of class C''.

Since (15) and (14) imply that $Y^{``} = \sigma^{-1}U_3'$, it follows from (5_3) that

$$(16) \qquad Y^{``} = -\sigma^{-1}\tau U_2,$$

hence $| Y'' | = | \tau/\sigma |$. This proves (11). But (11) implies, by (6), that $\kappa_0 \neq 0$. Consequently, there belongs to Γ_0: $Y = Y(s_0)$ three unit vectors, say $V_k = V_k(s_0)$, in the same way as the three unit vectors (2) belong to Γ: $X = X(s)$; so that

$$(17) \qquad V_1 = Y^{`}, \qquad V_2 = \kappa_0^{-1}Y^{``}, \qquad V_3 = [V_1, V_2].$$

But (16) and (11) show that $Y^{``} = \pm \kappa_0 U_2$, where

$$(18) \qquad \pm = \operatorname{sgn}(-\tau/\sigma).$$

If this and (15), respectively, are substituted into the second and the first of the relations (17), it follows that

$$(19) \qquad V_1 = U_3, \qquad V_2 = \pm U_2, \qquad -V_3 = \pm U_1,$$

since the matrices $\| U_1, U_2, U_3 \|$, $\| V_1, V_2, V_3 \|$ are orthogonal and of determinant $+1$.

Since every U_k is of class C' (as a matter of fact, of class $C^{4-2} = C''$) as a function of s, and since $s = s(s_0)$ is a function of class C' in s_0, it follows from (19) that all three functions $V_k = V_k(s_0)$ have continuous first derivatives $V_k^{`}$. In view of the criterion italicized at the beginning of Section 2, this proves that Γ_0 has a continuous torsion $\tau_0 = \tau_0(s_0)$.

Accordingly, only the explicit formula (12) remains to be proved. To this end, it is sufficient to apply (4) to the present case. In fact, this gives $\tau_0 = \det(V_1, V_2, V_2^{`})$, hence $\tau_0 = \det(U_3, U_2, U_2^{`})$, where $U_2^{`} = \sigma^{-1}U_2'$, by (14). It follows therefore from (5_2) that $\tau_0 = \det(U_3, U_2, -\sigma^{-1}\kappa U_1)$. In view of (4), this proves (12).

7. In view of the comment at the end of Section 4, it is worth pointing out that in the case

$$(20) \qquad \kappa(s) = \mathrm{k}, \quad \text{where } \mathrm{k} = \text{const.} > 0,$$

a case specifically discussed by Monge (cf. [1], pp. 43-44), the C^4-assumption of the last italicized theorem case be reduced to a C^3-assumption, and even to a C^2-assumption with a continuous torsion: Γ_0 *is of class* C'' *possessing a continuous curvature and a continuous non-vanishing torsion whenever* Γ *is of*

class C'' and possesses a constant curvature and a continuous non-vanishing torsion. In addition, (19) holds again, and the curvature $\kappa_0 = \kappa_0(s)$ and the torsion $\tau_0 = \tau_0(s)$ of Γ_0 are given, in terms of the curvature (20) and the torsion $\tau = \tau(s)$ of Γ, by

(21) $\kappa_0(s) = \mathrm{k}$, (22) $\tau_0(s) = \mathrm{k}^2/\tau(s)$.

The assumption (20) reduces the definition (9) to

$$\sigma = \tau/\mathrm{k}, \tag{23}$$

which in turn reduces (11)-(12) to (21)-(22). But this formal conclusion is not legitimate, since (23) depends on (9), and therefore on the C^4-assumption of Section 5-6. But if the continuous function $\sigma = \sigma(s)$ is *defined* by (23) under the assumption (20), then the last italicized statement follows by a straightforward repetition of the proof given in Section 6.

8. Let Γ: $X = X(s)$ be an arc of class C'' satisfying (1). Then there is at every point s of Γ an unique osculating circle. The latter has the radius $1/\kappa(s)$, and its center, say $Z = Z(s)$, is situated on the positively oriented principal normal $U_2 = U_2(s)$; so that

$$Z = X + \kappa^{-1} U_2. \tag{24}$$

Let Γ^* denote the locus of all points (24) when s varies on Γ. It will be assumed that Γ is sufficiently short.

If Γ is of class C''', then κ' exists and is continuous (cf. Section 3) and the vertices of Γ (if any) are defined as the points s at which κ becomes stationary, $\kappa' = 0$. Let it be assumed that either

$$\kappa' \neq 0 \tag{25}$$

or (6) holds, i. e., that τ and κ' do not vanish simultaneously. Then $Z' \neq 0$ holds at every point of Γ. In fact, if (24) is differentiated and X' and U_2' are then substituted from (2) and (5_2), it follows that

$$Z' = -\kappa'\kappa^{-2} U_2 + \tau\kappa^{-1} U_3. \tag{26}$$

Suppose in particular that Γ is a plane curve. Then (6) is violated identically and (26) reduces to

$$Z' = -\kappa'\kappa^{-2} U_2. \tag{27}$$

Since (2_2) and (1) show that the representation (24) of the evolute Γ^* of Γ is

$$\Gamma^*: Z = X + |X''|^{-2} X'', \tag{28}$$

two degrees of differentiability appear to be lost in the passage from Γ to Γ^*; so that Γ^* seems to be of class C' only, if Γ is just of class C'''. But it turns out that *one* of the two degrees of differentiability is *not* lost under the present assumptions, i. e., that the situation is as follows: *If a plane curve Γ of class C''' has a non-vanishing curvature and is free of vertices, then its evolute Γ^* is a curve of class C''.* In particular, Γ^* has a continuous curvature $\kappa^* = \kappa^*(s)$. The latter does not vanish, since its explicit representation proves to be the same as under the standard assumptions, namely, $\kappa^* = |\kappa'|/\kappa$.

All of this can be proved by adapting to (27) the procedures applied in Section 6. The proof will be omitted for this reason and also because the last italicized statement will follow in Sections 10-12 as a corollary of a more general theorem.

The cusps of the evolute of an ellipse show that the curve Γ^* need not be of class C' (and still less, as claimed by the assertion, of class C'') if the restriction (25) is omitted (if the ellipse is a circle, $\kappa' \equiv 0$, then Γ^* is not even a curve).

Actually, an easy calculation shows that if (25) is omitted from the assumptions of the last italicized theorem, and if P^* is the point of Γ^* corresponding to a point P of Γ, then Γ^* *must* have a cusp at P^* if κ' changes sign at P and does not vanish at points of Γ close enough to, but distinct from, P.

It should be noted that, in the last italicized statement, restriction of Γ to plane curves is essential, i. e., that *the curve* Γ^*: $Z = Z(s)$, defined by (24), need not be of class C'' if Γ: $X = X(s)$ is a twisted curve of class C''' which has a non-vanishing curvature and is free of vertices. It is understood that a vertex can be defined as a point s of Γ at which the derivative $Z'(s)$ of (24) vanishes, which, in view of (26), requires that neither of the conditions (6), (25) be satisfied. Actually, a counterexample can be chosen so as to satisfy both (6) and (25).

9. Let S: $X = X(u, v)$ be a surface of class C'''. Then the Gaussian and mean curvatures are functions, $K = K(u, v)$ and $H = H(u, v)$, of class C' satisfying the inequality $H^2 \geqq K$. Let S be free of umbilical points, that is, let

$$H^2 > K. \tag{29}$$

Then the principal curvatures k_1, k_2, being the roots of the quadratic equation $k^2 - 2Hk + K = 0$, are distinct and of class C'. Hence, at least one of these curvatures does not vanish (if (u, v) is confined to a sufficiently small

vicinity of any fixed (u^0, v^0)). The corresponding principal radius of curvature $\rho = 1/k$ is a function $\rho(u, v)$ of class C'.

The lines of curvature on S are defined, after a suitable rotation of the (u, v)-plane, by two differential equations of the form

$$dv/du = f(u, v), \tag{30}$$

where f is of class C', and consist of two transversal families, each of class C' and each covering S in a *schlicht* manner, if S is sufficiently small. By virtue of the equation of Rodrigues,

$$kdX + dN = 0, \tag{31}$$

there belongs to each principal curvature k *one* of these two families of curves, in the sense that (31) will hold along the curves of the family if sgn H, hence sgn k, and the orientation of the unit normal $N = N(u, v)$ are suitably chosen. If $k \neq 0$ and $\rho = 1/k$, then (31) can be written as

$$dX + \rho dN = 0. \tag{32}$$

While (32) is valid for at least one family of lines of curvature, that belonging to a non-vanishing principal curvature, it might not be possible to write the equation (31) corresponding to the other principal curvature in the form (32), since the vanishing (or even the identical vanishing) of the latter curvature is allowed, that is, the Gaussian curvature $K = k_1 k_2$ can satisfy $K(u, v) = 0$ (or even $K(u, v) \equiv 0$).

It will be assumed that sgn $\rho (\neq 0)$ and the orientation of N have been chosen so that (32) holds; in particular, the orientation of ρN is uniquely determined. The vector $Y = Y(u, v)$ defined by

$$Y(u, v) = X(u, v) + \rho(u, v) N(u, v) \tag{33}$$

is of class C'. The locus T: $Y = Y(u, v)$ need not be a surface of class C', since the condition

$$[Y_u, Y_v] \neq 0 \tag{34}$$

can be violated. However, it will be shown that if this condition is not violated, that is, *if* (33) *is a C'-parametrization of a surface T, then T must be a surface of class C''.* This does not mean, of course, that the *function* (33) is of class C'', but merely that the surface T, which is one of the two classical evolutes of the surface S (the evolute belonging to that root $\rho = \rho(u, v)$ of the quadratic [possibly linear] equation $K\rho^2 - 2H\rho + 1 = 0$ which occurs in the definition (33) of T) admits of *some* parametrization, say $T : Y = Y(u^*; v^*)$, in which it becomes of class C''.

Although (33) is a function of class C', the evolute T represented by it need not be a surface of class C', even if distinct points (u, v) correspond to distinct points $Y(u, v)$. This is shown by the example of a cylinder having an elliptical (not a circular) cross-section; cf. Section 10 below.

In order to assure that T is a surface of class C', it is sufficient to assume (29) and

$$\rho_\nu \neq 0, \tag{35}$$

where $\rho_\nu = \partial\rho/\partial\nu$ denotes the derivative of $\rho = \rho(u, v)$ in the direction of that line of curvature, passing through the point (u, v) of S, which is in the family of lines of curvature belonging to ρ by virtue of the normalization (32). In fact, (34) holds if and only if (29) and (35) hold (cf. [1], pp. 417-418). Accordingly, the last italicized statement is equivalent to the case $n = 3$ of the following theorem:

On a surface S: $X = X(u, v)$ of class C^n, where $n \geqq 3$, let (29) *hold and let $\rho = \rho(u, v)$ be a (finite) radius of principal curvature satisfying* (35). *Then the corresponding evolute T: $Y = Y(u, v)$, defined by* (35), *is a surface of class C^{n-1} (locally).*

This theorem is an analogue of theorem (iii) in [5], p. 368. The latter theorem replaces the evolute surface (33) by a parallel surface $Z = Z(u, v) = Z(u, v; r)$,

$$Z = X(u, v) + rN(u, v), \quad \text{where} \quad r = \text{const.} \gtrless 0, \tag{33*}$$

and claims that, except when the constant r satisfies the quadratic (possibly linear) equation $K(u, v)r^2 - 2H(u, v)r + 1 = 0$ at some point (u, v) of the (small) surface S, the parallel *surface* P_r: $Z = Z(u, v; r)$ is "smoother" than is indicated by the defining equation (33*). Curiously enough, the restriction placed on the value of the constant r, a restriction rendering P_r a surface of class C' (and playing, therefore, the same rôle as do the inequalities (29) and (35) above), is precisely $\rho(u, v) \neq r$, where $\rho(u, v)$ is a (finite) principal curvature, as in (33).

10. The content and the nature of the assertions of the last theorem can well be illustrated by the following corollary of it:

In an X-plane, where $X = (x, y)$, let S: $X = X(s)$ be an arc of class C^n, where s is arc length and $n \geqq 3$, and suppose that the curvature $\kappa = | X'' |$ $(= | N' |$, where N denotes $(-y', x')$, the normal vector) does not vanish and does not become stationary; that is,

(29 bis) $$\kappa(s) > 0$$
and
(35 bis) $$\kappa'(s) \neq 0.$$

Then the evolute T: $Y = Y(s)$, *defined by*

(33 bis) $$Y(s) = X(s) + \rho(s)N(s), \quad \text{where} \quad \rho = 1/\kappa,$$

is a curve of class C^{n-1} *(locally)*, even though the function $Y(s)$ is of class C^{n-2} only (unless $X(s)$ is of class C^{n+1}).

In fact, if V denotes the unit vector $(0, 0, 1)$ and if the plane vector $X = (x, y)$ is thought of as the vector $(x, y, 0)$, then $X(s, t) = X(s) + tV$ is a cylinder of class C^n. Condition (29 bis) assures that (29) holds on this surface; in fact, the principal curvatures k_1, k_2 on this cylinder are $k = \kappa > 0$ and $k_2 \equiv 0$. The family of lines curvature belonging to $\rho = 1/k_1 = 1/\kappa$ are the curves $t = \text{const.}$, that is, $X = X(s)$ and its congruent images $X = X(s) + tV$. Hence, condition (35) is equivalent to (35 bis). The evolute (33) belonging to the principal radius of curvature $\rho = 1/\kappa$ is the cylinder having the evolute (33 bis) as it cross-section; that is, the cylinder

$$T\colon\ Y(s, t) = Y(s) + tV \equiv X(s) + \rho(s)N(s) + tV.$$

11. In the following proof of the theorem italicized in Section 9, it will be assumed that $n = 3$. The proof will be such as to make clear its validity for $n > 3$ also.

Since S: $X = X(u, v)$ is of class C''' and (29) holds, a sufficiently small neighborhood of a point (u^0, v^0) can be transformed by a mapping $(u, v) \rightarrow (\alpha, \beta)$ of class C' and of non-vanishing Jacobian in such a way that the two families of the lines of curvature on S become represented by $\alpha = \text{const.}$ and $\beta = \text{Const.}$; cf. the above remarks concerning (30). Let $X = X(\alpha; \beta)$ denote the function which results if the functions $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$ of class C' are substituted into the function $X(u, v)$ (of class C'''). Thus $X = X(\alpha; \beta)$ is a C'-parametrization of the surface S (of class C'''), with the lines of curvature of S as parameter lines. Note that (as shown in [3], pp. 168-172) such a parametrization cannot, in general, be of class C'''; probably, it need not be even of class C''.

The normal $N(\alpha; \beta) = [X_\alpha, X_\beta]/|[X_\alpha, X_\beta]|$, which apparently is just continuous, is actually of class C' (as a function of (α, β)). For, on the one hand, $N(\alpha; \beta) \equiv \pm N(u, v)$ by virtue of $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$, while, on the other hand, $N(u, v)$ is a function of class C' (as a matter of fact, C'') as a

function of (u, v), and $u = u(\alpha, \beta)$, $v = v(\alpha, \beta)$ are functions of class C'. The principal curvatures $k_1(\alpha;\beta)$, $k_2(\alpha;\beta)$ and the principal radius of curvature $\rho(\alpha;\beta)$ are defined by invariance (for example, $\rho(\alpha;\beta) \equiv \pm\rho(u, v)$) and are therefore of class C'. The orientation of ρN is chosen so as to satisfy (32).

Although the (non-vanishing) perpendicular vectors X_α, X_β might only be continuous functions, the corresponding unit vectors $X_\alpha/|X_\alpha|$, $X_\beta/|X_\beta|$ are of class C' as functions of (α, β). In order to see this, note that $X_\alpha = X_u u_\alpha + X_v v_\alpha$. If (30) is the differential equation defining the lines of curvature $\beta =$ const., then $u_\alpha \neq 0$, and X_α is parallel to the non-vanishing vector $X_u + X_v(v_\alpha/u_\alpha) = X_u + X_v f(u, v)$. The latter is of class C' as a function of (u, v), and so the same is true of it as a function of (α, β). Consequently, $X_\alpha/|X_\alpha|$ (and similarly $X_\beta/|X_\beta|$) is of class C'.

Let α, β, $X(\alpha;\beta)$, $N(\alpha;\beta)$ be renamed u, v, $X(u, v)$, $N(u, v)$, respectively. Then $X = X(u, v)$ is just a C'-parametrization of the surface S of class C''', but $N(u, v)$ is of class C', and $u =$ const., $v =$ Const. are the two families of lines of curvature. Let the notation be so chosen that $v =$ Const. is the family which, in the sense specified after (32), belongs to the root $\rho = \rho(u, v)$ occurring in (33). Then (32) shows that the differential equation defining the lines of curvature $v =$ Const. is

$$X_u + \rho N_u = 0, \tag{36}$$

where, according to (35),

$$\rho_u \neq 0. \tag{37}$$

It also follows from (31) that, if $k_1 = k_1(u, v)$ and $k_2 = k_2(u, v)$ are the principal radii of curvature on S, and if the notation is so chosen that $\rho = 1/k_1$ in (36), then the equation (31) of the family $u =$ const. is

$$k_2 X_v + N_v = 0. \tag{38}$$

Finally, as verified above,

$$X_u/|X_u| \text{ is of class } C'. \tag{39}$$

12. Since all three functions X, N, ρ of (u, v) are of class C', the same is true of the function (33). But differentiation of (33) shows that

$$Y_u = \rho_u N \text{ and } Y_v = (1 - \rho k_2) X_v + \rho_u N \tag{40}$$

by virtue of (36) and (38), respectively. Hence it is easy to see that (34) holds.

In fact, the vector product of the derivatives (40) is

$$(41) \qquad [Y_u, Y_v] = \rho_u(1-\rho k_2)[N, X_v].$$

The vector $[N, X_v]$ does not vanish; in fact, since $u = \text{const.}$ and $v = \text{Const.}$ are lines of curvature, the non-vanishing vectors X_u, X_v, N are mutually perpendicular. Hence,

$$(42) \qquad [N, X_v] \neq 0 \text{ is parallel to } X_u/|X_u|.$$

The factor $1-\rho k_2$ in (41) is not 0, since the principal curvatures $k_1 = 1/\rho$, k_2 are distinct. Hence (34) follows from (41) and (37) and, since (41) and (42) imply that

$$(43) \qquad [Y_u, Y_v]/|[Y_u, Y_v]| = \pm X_u/|X_u|,$$

T is a surface of class C'.

In addition, (39) shows that T is a surface possessing a C'-parametrization $Y = Y(u, v)$ in which the unit normal (43) is of class C'. Hence, in order to conclude that, as claimed by the italicized theorem of Section 9, the surface T is of class C'', it is sufficient to apply the argument used at the end of Section 14 in [3], p. 163.

13. After the preceding generalization of the italicized statement of Section 8, a question complementary to that treated in Sections 4-6 will now be considered. In fact, whereas (10) was there assumed for $X = X(s)$ at every s, the identical violation of (10) will now be dealt with.

Assumption (10) for the function $\sigma = \sigma(s)$ defined by (9) is the natural condition for the non-degeneracy of the locus of the osculating spheres of a curve Γ: $X = X(s)$ and, correspondingly, the *identical* violation of (10) is the classical condition for a *spherical* Γ, that is, for a Γ satisfying $|X(s)| = \text{const.}$; cf. the footnote in Section 4. But this classical characterization of a Γ situated on a sphere (that is, the differential equation

$$(44) \qquad (\kappa^{-2}\kappa'\lambda)'\lambda = 1, \quad \text{where} \quad \lambda = \kappa/\tau,$$

which, in view of (9), is equivalent to $\sigma \equiv 0$) assumes, on the one hand, that κ and τ are of class C'' and C', respectively, implying for Γ an unnaturally strong C^4-restriction, and excludes, on the other hand, the cases of clustering or isolated zeros of the torsion (not to mention the case $\tau \equiv 0$ of a great circle Γ on the sphere $|X| = \text{const.}$); needless to say, $\tau(s) \not\equiv 0$ can have clustering zeros even if Γ is of class C^∞. In what follows, there will be derived a criterion which, though entirely explicit, is free of the artificial restrictions assumed in the classical criterion (44).

In order to simplify the formulae, suppose first that the radius of the sphere containing Γ: $X = X(s)$ is normalized to be 1,

$$(45) \qquad X^2 = 1.$$

If Γ is of class C'' (so that τ need not even exist), two differentiations of (45) give

$$(46) \qquad X \cdot U_1 = 0, \qquad X \cdot U_2 = -1/\kappa.$$

In fact, (2) is applicable under the C''-assumption alone, if the inequality (1) is satisfied. But it is, since the second derivative of (45) is $X'^2 + X \cdot X'' = 0$ which, in view of $|X'| = 1$, prevents the vanishing of X''. Actually, since X and U_2 are unit vectors, it follows from (46_2) that the inequality (1) can be improved to

$$(47) \qquad \kappa \geqq 1,$$

where the sign of equality holds if and only if $X \cdot U_2 = -1$ (which means that

$$(48) \qquad \kappa(s^0) = 1 \text{ if and only if } X(s^0) = -U_2(s^0)$$

at some $s = s^0$).

On the other hand, since X is a linear combination of the three vectors (2), it follows from (46_1) that there exist two (continuous) scalar functions $\alpha = \alpha(s)$, $\beta = \beta(s)$ satisfying

$$(49) \qquad X = \alpha U_2 + \beta U_3, \qquad \alpha^2 + \beta^2 = 1,$$

(49_2) being a consequence of (49_1) and (45) (since $|U_i| = 1$ and $U_2 \cdot U_3 = 0$). The coefficients of (49_1) are given by

$$(50) \qquad \alpha = -1/\kappa, \qquad \pm\beta = (1 - \kappa^{-2})^{\frac{1}{2}}; \quad \text{cf. } (47).$$

In fact, (49_1) and (46_2) imply (50_1), whence (50_2) follows by (49_2). The alternative sign in (50_2) remains undecided and can, by continuity, change only at points $s = s^0$ at which

$$(51) \qquad \beta(s^0) = 0, \quad \text{i.e.,} \quad \kappa(s^0) = 1 \text{ or } X(s^0) = -U_2(s^0);$$

cf. (50_2) and (48).

14. Needless to say, (49) and (50) are not only necessary but sufficient as well for a Γ of class C'' satisfying (45). It will now be assumed that such a Γ *satisfies the additional assumption of possessing a continuous torsion* $\tau = \tau(s)$ in the sense of Section 1. Under this assumption, it will be shown that

(I) *Γ must be of class C''' (i. e., κ' exists and is continuous)*;

(II) *$\kappa' = 0$ at all those points at which $\kappa = 1$*;

(III) *κ'' exists (and is non-negative) at all those points at which $\kappa = 1$*;

(IV) *the absolute value of $\tau = \tau(s)$ can be calculated from $\kappa = \kappa(s)$; in fact,*

$$\pm\,\tau = \kappa'/(\kappa^4 - \kappa^2)^{\frac{1}{2}} \text{ if } \kappa > 1, \tag{52}$$

$$\pm\,\tau = \kappa''^{\frac{1}{2}} \text{ if } \kappa = 1; \quad \text{cf. (III)}. \tag{53}$$

First, since Γ is supposed to have a continuous torsion, the functions U_i of s are of class C' (Section 2). Hence, scalar multiplication of (49_1) by U_2, U_3 proves the C'-character of a, β, respectively. It follows therefore from (50_1) that κ is a function of class C', and from (50_2), that the same is true of $\pm(\kappa^2 - 1)^{\frac{1}{2}}$ when the choice of the alternative sign is suitably made at points $s = s^0$ satisfying (48). The first of the latter two conclusions proves (I) if use is made of the italicized result of Section 3, while the second conclusion assures that, if $\kappa(s^0) = 1$ and $s^0 = 0$, then

$$\pm(\kappa^2 - 1)^{\frac{1}{2}} = cs + o(|\,s\,|) \text{ as } s \to 0 \tag{54}$$

holds for some constant c. The alternative sign in (54) is undecided and might change when s passes through 0. In any case $\kappa^2 - 1 = c^2s^2 + o(s^2)$, hence

$$\kappa = 1 + \tfrac{1}{2}c^2s^2 + o(s^2), \qquad \kappa' = o(1) \tag{55}$$

(in fact, (55_2) is implied by (55_1), since κ' exists and is continuous). Clearly, (55_1), (55_2) prove (III), (II), respectively.

Next, if the identity (49_1) is differentiated and X', U_2', U_3' are then substituted from (2_1), (5_2), (5_3), respectively, comparison of the coefficients of U_1, U_2, U_3 in the resulting identity supplies the three relations (50_1),

$$a' - \beta\tau = 0, \qquad \beta' + a\tau = 0. \tag{56}$$

But (50) and either (56_i) imply assertion (52) of (IV). Finally, assertion (53) of (IV) can be proved as follows:

According to (50_2), the function β is $1/\kappa$ times $\pm\,(\kappa^2 - 1)^{\frac{1}{2}}$. If this product is differentiated, it follows from (54) that

$$\beta' = -\,(cs + o(|\,s\,|)\kappa'/\kappa^2 + (c + o(1))/\kappa.$$

Since $\kappa \to 1$ as $s \to 0$, this implies that $\beta' \to c$, which, in view of (56_2), means that $a\tau \to -\,c$. It follows therefore from (50_1) that $\tau = c$ at $s = 0$, which, in view of (55), proves (53).

15. It is now easy to verify the following improvement of the C^4-criterion (44) (in a way which, in contrast to (44), does not exclude the important possibility $\tau = 0$): A Γ: $X = X(s)$ *of class* C'' *possessing a continuous torsion is a curve on the unit sphere if and only if it is of class* C''' *and its curvature and torsion satisfy* (47) *and* (52)-(53).

Needless to say, the normaliaztion of a spherical curve to be of radius 1 is unessential, since the theorem remains unaltered if τ and κ in (52)-(53) are replaced by $r\tau$ and $r\kappa$, respectively, when (45) is replaced by $X^2(s) \equiv r^2$, where r is any positive constant.

It is clear from (I)-(IV), Section 14, that the last italicized theorem will follow if it is ascertained that a Γ: $X = X(s)$ of class C''' must satisfy (45) whenever its curvature and torsion are subject to (47) and (52)-(53). But (5) and the assignment of any pair of continuous functions $\tau(s), \kappa(s) > 0$ determine for (5) a solution (U_1, U_2, U_3) of orthonormal vectors (of determinant $+1$) which is unique modulo the group of euclidean movements. On the other hand, if $\tau(s)$ is given, in terms of a $\kappa(s)$, by (52)-(53), then, by retracing the steps made in Sections 13-14, it can be verified that the $X = X(s)$ defined by (49) and (50) must satisfy (2) and (5). This proves the last italicized statement.

16. In the direction of the results of this paper, all of which deal with the possibility of savings in the assumed degree differentiability, a final remark will now be made on the Legendre transformation of a curve Γ: $y = y(x)$ in an (x, y)-plane, that is, *on the replacement of the point coordinates of a plane curve by its line coordinates.* Formally, this consists of the replacement of x, y by ξ, η, where, if $' = d/dx$,

(57$_1$) $\qquad \xi = y', \qquad\qquad$ (57$_2$) $\qquad \eta = xy' - y,$

and of the assertion that the inverse of the substitution $(x, y) \rightarrow (\xi, \eta)$ is

(58$_1$) $\qquad x = \eta^{\backprime}, \qquad\qquad$ (58$_2$) $\qquad y = \xi\eta^{\backprime} - \eta,$

where $^{\backprime} = d/d\xi$.

In order to formulate conditions under which this formalism is valid, it must first be assumed that $y(x)$ has on some x-interval (a, b) a derivative $y'(x)$ (in some sense) and that, if (57$_1$) maps (a, b) onto the ξ-interval (α, β), the functions (57$_i$), (58$_i$) are considered on (a, b), (α, β), respectively. Then the classical theorem on Legendre's transformation can be

formulated as follows (cf. [6], pp. 6-8, where $x, y; \xi, \eta$ are vectors with n components; so that the present case results by choosing $n = 1$ *loc. cit.*):

Suppose that $y(x)$ is of class C'' and that, in addition, its second derivative (the "Hessian" of $y(x)$, i. e., the "Jacobian" of (57_1)) is subject to the restriction

$$(59) \qquad y'' \neq 0$$

on (a, b). Then it is clear that (57_1) maps (a, b) onto (α, β) in a one-to-one C'-manner and in such a way that the inverse mapping $\xi \rightarrow x$, too, is of class C'. The classical theorem states that if this C'-function $x = x(\xi)$ is substituted into the C'-function (57_2) of x, the resulting function $\eta = \eta(\xi)$ will be of class C'' on (α, β), and that (58_1)-(58_2) is the explicit form of the inverse of the mapping (57_1)-(57_2) of (x, y) onto (ξ, η).

Notice however that the C''-assumption on $y(x)$ and the restriction (59) are only sufficient in order that the mapping (57_1) of (a, b) onto (α, β) be continuous and one-to-one, i. e., for the following condition:

$$(60) \qquad y'(x) \text{ is strictly monotone} \qquad ('= d/dx)$$

on (a, b). In fact, if

$$(61_1) \qquad y(x) = |\, x^3 \,|^{\frac{1}{2}}, \qquad\qquad (61_2) \qquad y(x) = x^4,$$

then both functions (61_i) are of class C' and such as to satisfy (60) but, if $a < 0 < b$, the function (61_1) fails to be of class C'' while the function (61_2) fails to satisfy (59). These examples reveal the content of the following extension (applicable to both (60_1) and (60_2)) of the classical theorem:

If $y(x)$ is a differentiable function satisfying (60) *on* (a, b) (*which implies that $y(x)$ is of class C'*), *and if the inverse $x = x(\xi)$ of the function* (57_1) *of x is substituted into the function* (57_2) *of x, then the resulting function $\eta = \eta(\xi)$ is differentiable on the ξ-interval* (α, β) *which corresponds to* (a, b), *and*

$$(60 \text{ bis}) \qquad \eta^{\grave{}}(\xi) \text{ is strictly monotone} \qquad (\grave{} = d/d\xi)$$

on (α, β) (*which implies that η is of class C' as a function of ξ*); *finally, the explicit form of the inverse of the contact transformation* $(x, y) \rightarrow (\xi, \eta)$, *defined by* (57_1)-(57_2), *is given by* (58_1)-(58_2).

The proof of this somewhat unexpected extension of the classical theorem depends only on careful applications of the definition ($= \lim \Delta v/\Delta u$) of a derivative, and will be omitted. Actually, this theorem is just a degenerate case (dealing with plane curves) of Theorem (II) (dealing with surfaces) in a paper of Hartman and myself, to appear in this JOURNAL.

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ON THE EXISTENCE OF RIEMANNIAN MANIFOLDS WHICH CANNOT CARRY NON-CONSTANT ANALYTIC OR HARMONIC FUNCTIONS IN THE SMALL.*

By Philip Hartman and Aurel Wintner.

A function $f = f(x, y)$ on an open domain D of the real (x, y)-plane is said to be of class $C^n(\lambda)$, where $n \geqq 0$ and $0 \leqq \lambda \leqq 1$, if all partial derivatives $\partial^n f/\partial^m x \partial^{n-m} y$, where $0 \leqq m \leqq n$, exist and are continuous on D and satisfy a uniform Hölder condition of index λ on every compact subset of D. If the assumption of such an index λ is omitted, then $C^n(\lambda)$ reduces to the class $C^n = C^n(0)$. In particular, $C^0(1)$ is the class of functions satisfying a locally uniform Lipschitz condition, and $C^0 = C^0(0)$ is the class of all continuous functions, on D. In the questions to be dealt with below, there is no loss of generality in assuming that D is *schlicht*, simply connected and small, say

$$(1) \qquad D_a\colon\ x^2 + y^2 < a^2,$$

where $a > 0$ is arbitrarily fixed.

If $g_{11}, g_{12} = g_{21}, g_{22}$ are three real-valued functions of class $C^n(\lambda)$ on D and have the property that the matrix (g_{ik}) is positive definite at every point of D, then

$$(2) \qquad ds^2 = g_{11}dx^2 + 2g_{12}dxdy + g_{22}dy^2, \quad \text{where} \quad g_{ik} = g_{ik}(x, y),$$

will be called a $C^n(\lambda)$-metric (on D). It will be referred to as a C^n-metric if $\lambda = 0$, and as a continuous metric if it is a C^0-metric. If a in (1) is small enough and if two functions

$$(3) \qquad u = u(x, y), \qquad v = v(x, y)$$

are of class $C^{n+1}(\lambda)$, have a non-vanishing Jacobian on D_a and are normalized by

$$(4) \qquad (u(0,0), v(0,0)) = (0, 0),$$

then (3) has a unique inverse

$$(5) \qquad x = x(u, v), \qquad y = y(u, v),$$

* Received October 22, 1952.

where both functions (5) are of class $C^{n+1}(\lambda)$ on the neighborhood E_a of the point (4), if E_a denotes the image of D_a in the (u, v)-plane (and

$$(6) \qquad (x(0,0), y(0,0)) = (0,0)$$

is the image of the point (4) under the mapping (5)). Clearly, any such $C^{n+1}(\lambda)$-mapping transforms every $C^n(\lambda)$-metric (2) on D_a into a $C^n(\lambda)$-metric on E_a, say into

$$(7) \qquad ds^2 = h_{11}du^2 + 2h_{12}dudv + h_{22}dv^2, \quad \text{where} \quad h_{ik} = h_{ik}(u, v),$$

the coefficient matrix of (7) being given in terms of that of (2) by the requirement that

$$(8) \qquad (h_{ik}) = J(g_{ik})J' \text{ by virtue of (3) or (5)}$$

(in the sense of matrix multiplication), where J' denotes the transposed matrix of the Jacobian matrix J of (3) (in particular,

$$(9) \qquad \det h_{ik} = (\det J)^2 \det g_{ik},$$

where $\det g_{ik} > 0$ and $\det J \neq 0$). Conversely, if (2) and (7) are two $C^n(\lambda)$-metrics which become identical by virtue of a C^1-transformation (5) of non-vanishing Jacobian, then this $C^1(0)$-transformation must be of class $C^{n+1}(\lambda)$. This was proved in [5] for $\lambda = 0$ only, but the proof given there is valid for every λ.

Two continuous metrics, say (2) and (7), given on respective vicinities of points (6) and (4), are called isometric if these vicinities can be transformed into one another by C^1-transformations, (3) and (5), by virtue of which the two metrics become identical. Clearly, a continuous metric (7) of arbitrarily smooth coefficient functions (even the Euclidean metric $du^2 + dv^2$) is isometric with certain C^0-metrics (2) which are not C^1-metrics. It is therefore natural to consider the class, say C_0, of those continuous metrics, the "exactly continuous metrics," which are not isometric with any C^1-metric and, more generally, the class, say C_n, of those C^n-metrics which are not isometric with any C^{n+1}-metric.* The existence of an "exactly continuous metric" is

* There is a corresponding question for pairs of linear, instead of quadratic, differential forms, say

$$(2') \quad a_1(x, y)\,dx + a_2(x, y)\,dx, \qquad (7') \quad b_1(u, v)\,du + b_2(u, v)\,dv,$$

which are isometric in the sense that (2′) is identical with (7′) by virtue of some C^1-transformation (3) of non-vanishing Jacobian. Let a Pfaffian (2′) be called of class C^n if its coefficients $a_i(x, y)$ are of class C^n, and let (2′) be called of class C_n if it is of class C^n but is not isometric to any Pfaffian (7′) of class C^{n+1}. The existence of

not obvious at all. That such metrics exist will follow, as a corollary, from Theorem (i) below, if the latter is combined with the parenthetical assertion of Theorem (ii) below. Similarly, Theorem (ii_n) below, when combined with (ii_{n+1}), implies that the class C_n is not vacuous for any n.

If (2) is a continuous metric (on D_a) corresponding to which a transformation (3) (of D_a into an E_a) can be so chosen that h_{11} and h_{12} in (7) become identical and h_{12} becomes identically 0, then (2) is said to possess a conformal normal form (on D_a). The latter is characterized by the existence of a continuous function h satisfying

$$ds^2 = h^2 \cdot (du^2 + dv^2), \tag{10}$$

where $h = \pm(\det g_{ik})^{\frac{1}{2}}\partial(u, v)/\partial(x, y)$ in view of (9), hence

$$h = h(u, v) > 0 \tag{11}$$

without loss of generality.

(i) *If* (2) *is a continuous metric on a circle* (1), *then, no matter how small the latter be chosen, there need not exist any transformation* (3), *of class* C^1 *and of non-vanishing Jacobian, which transforms* (2) *into a conformal normal form* (10).

The question as to the existence of such metrics (2) was raised in [14], p. 203, where it was shown (pp. 204-205) that the answer is in the negative if (2), instead of being definite as above ($\det g_{ik} > 0$), is indefinite ($\det g_{ik} < 0$) and, correspondingly, (10) is replaced by $ds^2 = h^2 \cdot (du^2 - dv^2)$. It should be noted that C^1-metrics cannot be admitted in (i); what is more, such metrics as comply with (i) cannot be $C^0(\lambda)$-metrics for any $\lambda > 0$. In fact, if $0 < \mu < \lambda \leqq 1$, then, according to Lichtenstein [10], there belongs to every $C^0(\lambda)$-metric (2) a transformation (3) which is of class $C^1(\mu)$ (hence of class C^1) along with its inverse (5), and transforms (2) into a conformal normal form (10). Thus the point of (i) is that Lichtenstein's arbitrarily small $\lambda > 0$ cannot be replaced by $\lambda = 0$.

Since every C^1-solution $u^* = u^*(u, v)$, $v^* = v^*(u, v)$ of the Cauchy-Riemann equations $u^*_u = v^*_v$, $u^*_v = - v^*_u$ is analytic, it is clear that if a continuous metric (2) has a conformal normal form in terms of the parameters u, v, then it will have a conformal normal form in some other parameters u^*, v^* if and only if $w^* = u^* + iv^*$ is an analytic function of

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Pfaffians of class C_n is obvious only if $n > 0$. That $n = 0$ need not actually be excluded (i. e., that there exist "exactly continuous" Pfaffians), can readily be concluded from the result italicized in [14], Section 7, pp. 205-206.

$w = u + iv$ satisfying $dw^*/dw \neq 0$. But this universality (with regard to the choice of (2)) of the analytic functions of a complex variable depends on the assumption that (2) has *some* conformal normal form. And it turns out that there exist continuous metrics (2) for which this assumption is not satisfied in the neighborhood of any point; in other words, that *a positive definite, continuous, Riemannian metric* (2) *on the circle* $D_a: x^2 + y^2 < a^2$ *need not carry any non-constant analytic function* (on D_b, no matter how small $b < a$ be chosen).

The Cauchy-Riemann equations belonging to a continuous metric (2) are

$$(12) \qquad gv_x = g_{12}u_x - g_{11}u_y, \qquad gv_y = g_{22}u_x - g_{12}u_y$$

(Riemann, Beltrami; cf. [8], pp. 520-521, [1], pp. 126-127), where

$$(13) \qquad g = (\det g_{ik})^{\frac{1}{2}} > 0.$$

But (12) implies that the Jacobian $u_x v_y - u_y v_x$ cannot vanish identically unless

$$(14) \qquad u = \text{const.}, \qquad v = \text{Const.}$$

Hence it is clear that the last italicized statement is the substance of the following refinement of (i):

(i*) *There exist continuous metrics* (2) *for which the system* (12) *does not possess any solution distinct from* (14), provided that by "a solution (3)" is meant a pair of functions (3) for which the partial derivatives

$$(15) \qquad u_x, u_y; \qquad v_x, v_y$$

exist, satisfy (12) and are *continuous.*

Beltrami's first differential parameter belonging to (2) is $\nabla(u, u)$, where, if g is defined by (13),

$$(16) \qquad g^2 \nabla(u, z) = g_{22}u_x z_x - g_{12}(u_x z_y + u_y z_x) + g_{11}u_y z_y$$

(cf. [1], pp. 76-77). Hence Dirichlet's problem belonging to a continuous metric (2) is

$$(17) \qquad \min \iint_{[\Gamma]} g\nabla(u, u)\,dxdy \text{ when } u(x, y) = \phi \text{ on } \Gamma.$$

Here $[\Gamma]$ denotes the interior of a Jordan curve Γ contained in the domain (1) on which (2) is given as a continuous metric, and ϕ is a preassigned continuous function of position on Γ. What is sought for is a function $u(x, y)$ which is continuous on $[\Gamma] + \Gamma$, of class C^1 on $[\Gamma]$, equal to ϕ on Γ, and

such as to minimize the integral (17) with reference to all such functions $u(x,y)$. In view of Hilbert's method ([7], pp. 10-14 and pp. 15-37; cf. in particular the general comments referred to in the footnote on p. 11 and the italicized statement to which it belongs on p. 11), it seems to be of methodical interest that (i*) implies (and is substantially equivalent to) the following negative result:

(i bis) *There exist on* (1) *continuous metrics* (2) *corresponding to which the Dirichlet problem* (17) *has on* $[\Gamma]$ *no solution* $u(x,y)$ *of class* C^1 (*and continuous on* $[\Gamma]+\Gamma$) *with reference to any Jordan curve* Γ *contained in* (1) *and to any continuous non-constant boundary function* ϕ (if $\phi=$ const. on $[\Gamma]$, then $u(x,y)=$ const. is of course a solution of (17) on $[\Gamma]+\Gamma$).

It is essential that in this restatement (i bis) of (i*) no (x,y)-set of measure 0 is excluded from $[\Gamma]$; in this regard, cf. [12].

In order to prove (i bis), choose (2) as in (i*), suppose that the assertion of (i bis) is false for some Γ and some ϕ ($\neq$ const.) on Γ and denote by $u(x,y)$ the (or a) corresponding solution of (17). Then, if $z=z(x,y)$ is any function of class C^1 on $[\Gamma]+\Gamma$ satisfying $z\equiv 0$ on Γ, the value of the Dirichlet integral (17) is not less for $u+z$ than for u itself. Hence

$$\iint_{[\Gamma]} g\nabla(u,z)\,dudv=0 \tag{18}$$

follows in the usual way (that is, from the bilinear character of the operator (16) and from the fact that the matrix of the bilinear form (16), being g times the matrix of (2), is positive definite). Insertion of (16) into (18) gives

$$\iint_{[\Gamma]} (az_x+bz_y)\,dxdy=0 \tag{19}$$

if $a=a(x,y)$ and $b=b(x,y)$ are defined by

$$ga=g_{22}u_x-g_{12}u_y,\qquad gb=g_{11}u_y-g_{12}u_x. \tag{20}$$

Hence the functions a, b are continuous on the open set $[\Gamma]$ and, in view of the finiteness of the integral (17), Schwarz's inequality implies that a, b are of class L^2 (and therefore absolutely integrable) on $[\Gamma]$ or, since meas $\Gamma=0$, on $[\Gamma]+\Gamma$. It follows therefore from (a trivial extension of) Haar's lemma [3], p. 2 (where a, b are supposed to be continuous on $[\Gamma]+\Gamma$), that the truth of (19) for each of the above-mentioned functions $z=z(x,y)$ implies the vanishing of

$$\int (ady-bdx), \tag{21}$$

where the integration path is any rectifiable Jordan curve contained in $[\Gamma]$. This means that the integral (21), when extended within $[\Gamma]$ from a fixed point (x_0, y_0) to a variable point (x, y), is a point function, say $v = v(x, y)$. But the function thus defined on $[\Gamma]$ has the partial derivatives $v_x = -b$, $v_y = a$, by (21). In view of (20) and (13), this means that $v(x, y)$ is of class C^1 on $[\Gamma]$ and satisfies (12). If this is compared with (i*), the assertion of (i bis) follows.

Theorem (i) will be paralleled by the following:

(ii) *If* (2) *is a C^1-metric on a circle* (1), *then there* (*exists a transformation* (3) *of class C^1 but*) *need not exist any transformation* (3) *of class C^2, of non-vanishing Jacobian, which transforms* (2) *into a conformal normal form* (10).

As mentioned after (i), the parenthetical (positive) assertion of (ii) is a corollary of Lichtenstein's theorem. In view of Riemann's mapping theorem and of the remarks made before (i), this assertion of (ii) is equivalent to the statement that all functions analytic on a circle of the ordinary complex $(u + iv)$-plane can be transferred to the circle (1) so as to become analytic functions with reference to the C^1-metric (2). In fact, the Cauchy-Riemann equations (12) are then satisfied. But the main (negative) assertion of (ii) is that the functions (3), which represent the real and imaginary parts of the analytic functions on (2), will become of class C^2 only in the trivial case (14), if the C^1-metric (2) is suitably chosen.

The analogue of the refinement (i*) of (i) is the following:

(ii*) *There are C^1-metrics on which there does not exist any non-constant harmonic function* (*although all analytic functions exist on every C^1-metric*), provided that by an harmonic function of a C^1-metric (2) is meant a function $u = u(x, y)$ for which the partial derivatives

$$(22) \qquad u_x, u_y, u_{xx}, u_{xy}, u_{yx}, u_{yy}$$

exist, are *continuous* and such as to satisfy the condition expressed by the identical vanishing of Beltrami's second differential operator, that is, by the partial differential equation

$$(23) \qquad \{(g_{22}u_x - g_{12}y_y)/g\}_x + \{(g_{11}u_y - g_{12}u_x)/g\}_y = 0;$$

cf. [1], p. 127. Note that (23) is the (formal, whereas

$$\int_{\Gamma} (ady - bdy) = 0 \tag{21 bis}$$

with (20) is the unrestricted) integrability condition ($v_{xy} = v_{yx}$) of (12).

What belongs to (ii) in the same way as (i bis) belongs to (i) is the following circumstance:

(ii bis) *There exist on* (1) *metrics* (2) *of class* C^1 *corresponding to which the Dirichlet problem* (17) *belonging to a smooth Jordan curve* Γ *and a smooth boundary function* ϕ *can have a solution* $u(x, y)$ *of class* C^1, *although the Euler-Lagrange equation of* (17) *fails to have a non-constant solution of class* C^2.

In fact, the latter equation is (23), while the system of Haar ([3], pp. 16-17) belonging to (17) reduces to (12). Hence the last italicized statement follows from (ii*). A corresponding situation for the hyperbolic Euler-Lagrange equation $u_{xx} - u_{yy} = 0$ was pointed out by Hadamard [7], pp. 242-243, and for the elliptic (but inhomogeneous) Euler-Lagrange equation $u_{xx} + u_{yy} = f(x, y)$, where $f(x, y)$ is continuous, by Lichtenstein [9] (a corresponding example for the homogeneous equation $u_{xx} + u_{yy} + f(x, y)u = 0$ follows from [13], p. 733).

Both (i) and (ii) will be proved by choosing a suitable positive function $g = g(x, y)$ and placing $g_{11} = 1$, $g_{12} = 0$, $g_{22} = g^2$. Then (2) becomes

$$ds^2 = dx^2 + g^2dy^2, \qquad (g > 0), \tag{24}$$

the g occurring in (24) is identical with the square root (13). The Cauchy-Riemann equations (12) simplify to

$$v_x = - g^{-1}u_y, \qquad v_y = gu_x \tag{25}$$

if (24) is a C^0-metric (i. e., if g is continuous), and the Laplace equation (23) can be replaced by

$$(gv_x)_x + (g^{-1}v_y)_y = 0 \tag{26}$$

if (24) is a C^1-metric (i. e., if g is a function of class C^1). In fact, if the two linear equations (12) are solved with respect to u_x, u_y and, correspondingly, the integrability condition $v_{xy} = v_{yx}$ of (12), which is (23), is replaced by $u_{xy} = u_{yx}$, then what results in the case (24) of (2) is (26).

It will be clear from the proof of (ii) that (ii) can be generalized as follows:

(ii_n) *The assertions of* (ii) *remain true if the classes* C^1, C^2 *referred to in* (ii) *are replaced by* C^n, C^{n+1}, *respectively, where* n *is any positive integer.*

In view of (i), this holds for $n=0$ also, except that what then corresponds to the parenthetical remark of (ii) must be omitted as meaningless (in fact, the functions (5) cannot be substituted into (2) if they are just of class C^0, i. e., continuous).

In the proof of (i), the following lemma (†) on inhomogeneous Cauchy-Riemann equations will be needed:

(†) *If* λ, μ *is a given pair of continuous functions on a circle*

$$D_a\colon\ x^2+y^2<a^2,$$

then the system

$$u_x - v_y = \lambda(x,y), \qquad u_y + v_x = \mu(x,y)$$

cannot possess any solution $u=u(x,y)$, $v=v(x,y)$ *of class* C^1 *on any circle* D_b *unless*

$$\lim_{\epsilon\to+0}\int_0^{2\pi}\Big(\int_\epsilon^b r^{-1}(\lambda\cos 2\theta+\mu\sin 2\theta)\,dr\Big)d\theta$$

exists as a finite limit (for every and/or some, sufficiently small, value $b>0$ *of the upper limit of integration), where*

$$\lambda=\lambda(r\cos\theta, r\sin\theta), \qquad \mu=\mu(r\cos\theta, r\sin\theta).$$

It will be convenient to prove the above statements in the following order: (ii), (i), (†), (ii*), (i*).

Proof of (ii). Let $a<1$ in (1) and define on (1) a continuous function h by placing

(27) $$h(x,y)=x^2/(r^2\log r^2),\ \text{if}\ 0<r<a,\ \text{and}\ h(0,0)=0,$$

where $r=(x^2+y^2)^{\frac12}\geqq 0$ (this is the function used by Petrini [11], p. 138, to show that Poisson's equation

(28) $$v_{xx}+v_{yy}=f(x,y)$$

need not have a solution $v(x,y)$ of class C^2 if $f(=h)$ is just continuous). It turns out (cf. [6], p. 136) that the function

(29) $$g(x,y)=1+\int_0^x h(t,y)\,dt$$

is of class C^1. Clearly, it is positive on (1) if a is sufficiently small. Thus (24) is a C^1-metric on the circle $D_a: x^2 + y^2 < a^2$ if a is small enough.

In order to prove (ii), it will first be shown that the case (29) of the Laplace equation (26) cannot have on D_a (for any sufficiently small a) any solution $v(x, y)$ of class C^2 satisfying

$$(30) \qquad v_x(0,0) = 1 \quad \text{and} \quad v_y(0,0) = 0.$$

This will be concluded by adapting the arguments used in [13], pp. 736-738 as follows:

Let $v = v(x, y)$ be any function of class C^2 satisfying (26). Then, for this v, (15) can be written in the form of a Poisson equation (28), in which $f = f(x, y)$ is

$$(31) \qquad f = f_1 + f_2 + f_3 + f_4,$$

if the four functions $f_j = f_j(x, y)$ are defined by

$$(32_1) \quad f_1 = (1 - g)v_{xx}; \qquad (32_2) \quad f_2 = (1 - g^{-1})v_{yy};$$

$$(32_3) \quad f_3 = g^{-2}g_y v_y; \qquad (32_4) \quad f_4 = - g_x v_x.$$

But if $f(x, y)$ is any function continuous (and, say, bounded) on a circle (1), then results of Zaremba [15] imply (cf. [13], p. 735) that the corresponding Poisson equation (28) has on (1) no continuous solution v (that is, there exists on (1) no continuous v possessing second derivatives ϕ_{xx}, ϕ_{yy} the sum of which is f) unless such a v is represented by the logarithmic potential, say $v^* = v^*(x, y)$, belonging to the density $(2\pi)^{-1}f(x, y)$; in which case every such v is of the form $v = v^* + w$, where w is a regular harmonic function on (1) (in the sense of the euclidean metric, i. e., $w_{xx} + w_{yy} = 0$). On the other hand, Petrini has shown ([11], pp. 131-134) that the partial derivative v^*_{xx} and/or v^*_{yy} of the logarithmic potential $v^*(x, y)$ will exist at $(x, y) = (0, 0)$ if and only if

$$(33) \qquad \lim_{\epsilon \to +0} \int_0^{2\pi} \left(\int_\epsilon^a r^{-1} f(r \cos \theta, r \sin \theta) dr \right) \cos 2\theta d\theta \quad \text{exists}$$

(as a finite limit) for the function $f = f(x, y)$ defining v^* and occurring in (28). Hence, if (33) is satisfied by each of the three functions f_1, f_2, f_3 but is not satisfied by f_4, then the case (31) of (28) cannot have any solution v of class C^2 on (1), which means that the same is true of (26). Consequently, in order to prove that (26) has no solution v of class C^2 satisfying (30), it is

sufficient to show that the first three of the functions (32_1)-(32_4) which belong to such a v must, but the fourth of them cannot, satisfy condition (33).

First, since the function (29) is of class C^1 and becomes 1 at $(x,y) = (0,0)$, it is clear that $1 - g = O(r)$, where $r = (x^2+y^2)^{\frac{1}{2}} \to 0$. Hence (32_1), (32_2) and the continuity (in fact, just the boundedness) of v_{xx}, v_{yy} imply that $f_1 = O(r)$, $f_2 = O(r)$. Consequently, (33) is satisfied by $f = f_1$ and by $f = f_2$. On the other hand, since v is of class C^2, hence v_y of class C^1, the second of the assumptions (30) implies that $v_y = O(r)$. It follows therefore from (32_3), where $g \to 1$ and $g_y \to g_y(0,0)$ as $r \to 0$, that $f_3 = O(r)$, and so (33) is satisfied by $f = f_3$ also. It remains to show that (33) is violated by $f = f_4$.

To this end, note that, since v_x is of class C^1, the first assumption in (30) and the definition (32_4) imply that $f_4 = (-1 + O(r))g_x$. It follows therefore from (29) that $f_4 = -h + O(r)$. Hence $f = f_4$ violates (33) if $f = h$ does. But the function $f = h$, defined by (27), is precisely Petrini's example of a continuous function violating (33); cf. [11], p. 138.

This proves that (26) cannot have a solution v of class C^2 satisfying (30). Hence, in order to prove (ii), it is sufficient to ascertain that the negation of the statement of (ii) implies the existence of such a v. Suppose therefore that (ii) is false with reference to the C^1-metric (24), defined by (29). Then, if a is small enough, there exists for the circle (1) a transformation (3) which is of class C^2 along with its inverse (5) and which transforms (24) into $du^2 + dv^2$ times a positive function of (u, v) (in view of (8), this positive function is of class C^1). Hence the C^2-mapping (3) transforms every function $t = t(u,v)$, which is a regular harmonic function (i. e., function of class C^2 satisfying the Laplace equation $t_{uu} + t_{vv} = 0$) in a vicinity of the point (4), into a function

$$\tau(x,y) \equiv t(u,v) \tag{34}$$

which, in a vicinity of the point (6), is of class C^2 and a solution $v = \tau$ of the Laplace equation (26) (the v in (26) will not be confused with the v in (3) or (5)). Since the regular harmonic function $t(u,v)$ is arbitrary in (34), and since the Jacobian of (3) does not vanish, it is clear that a $\tau(x,y)$ can be so chosen that the initial values $\tau_x(0,0)$, $\tau_y(0,0)$ of its first partial derivatives become preassigned numbers. Consequently, both (26) and (30) can be satisfied by a function $v = \tau(x,y)$ of class C^2. This contradiction completes the proof of (ii).

Proof of (i). Let $a < 1$ in (1) and define the continuous function h on D_a again by (27) (that is, by

$$h(x,y) = \tfrac{1}{2}(1+\cos 2\theta)/\log r^2, \quad \text{where} \quad h(0,0) = 0 \tag{35}$$

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and $x = r\cos\theta$, $y = r\sin\theta$), but let (29) now be replaced by

$$(36) \qquad g(x,y) = 1 + h(x,y) \qquad (g(0,0) = 1).$$

Then, if a is small enough, $g(x,y)$ is positive, hence (24) is a continuous metric, on D_a. It will be shown that this metric has the property claimed by (i).

To this end, it will first be shown that the case (36) of the system (25) cannot possess on D_a any solution $u = u(x,y)$, $v = v(x,y)$ of class C^1 satisfying

$$(37) \qquad u_x(0,0) = 1, \qquad u_y(0,0) = 0.$$

In order to prove this, let u, v be any pair of functions which are of class C^1, and satisfy (25) identically in (x,y), on D_a. These two (x,y)-identities can be written in the form $u_x - v_y = \lambda$, $u_y + v_x = \mu$, where $\lambda = \lambda(x,y)$, $\mu = \mu(x,y)$ denote the functions defined by

$$(38) \qquad \lambda = (1-g)u_x, \qquad \mu = (1-g^{-1})u_y;$$

functions which are continuous on D_a, since u_x, u_y and $g(>0)$ are. It follows therefore from the statement of Lemma (†), italicized before (27), that the system (25) cannot have any solution of class C^1 satisfying (37) if the limit of the double integral, occurring in (†), fails to exist (as a finite limit) for the functions (38). Hence it is sufficient to show that the condition

$$\int_0^{2\pi} \left(\int_\epsilon^b r^{-1}(\lambda\cos 2\theta + \mu\sin 2\theta)\,dr\right) d\theta \to \infty, \text{ as } \epsilon \to +0,$$

is satisfied by the functions (38) and by some and/or all, sufficiently small, value of $b > 0$. In particular, it is sufficient to show that the integral is of the form

$$\int_\epsilon^b (r\log r^2)^{-1}(-\tfrac{1}{2}\pi + o(1))\,dr,$$

where the $o(1) = o(r)/r$ refers to $r \to 0$. Consequently, it is sufficient to ascertain that, if $x = r\cos\theta$ and $y = r\sin\theta$ in $\lambda = \lambda(x,y)$, $\mu = \mu(x,y)$, and if $r \to 0$, then, uniformly in θ,

$$\lambda = -\tfrac{1}{2}(1+\cos 2\theta)L + o(|L|) \text{ and } \mu = o(|L|), \text{ where } L = (\log r^2)^{-1}.$$

But the latter pair of relations is readily verified. In fact, (37) means that the (continuous) functions u_x, u_y are of the respective forms $1 + o(1)$, $o(1)$.

It follows therefore from (38) and (36) that λ is $-h+ho(1)$ and that μ is $1-(1+h)^{-1}$ times $o(1)$. Hence the pair of relations claimed by the last formula line follows from (35). This proves that (25) cannot have any solution of class C^1 satisfying (37).

In order to conclude from this that the continuous metric (24) defined by (36) has the property claimed in (i), suppose that it does not. The transformation supplied by this negation, a transformation of class C^1 and of non-vanishing Jacobian, can be used in the same way as, at the end of the proof of (ii), the corresponding transformation of class C^2 was used in conjunction with (34). In fact, it is sufficient to replace there the (real) harmonic functions t, occurring in (34), by the analytic functions $s+it$ which are regular at the origin of a complex plane. By virtue of the C^1-transformation of non-vanishing Jacobian, the real and imaginary parts of all these regular function elements become functions $u=u(x,y)$, $v=v(x,y)$ of class C^1 satisfying the Cauchy-Riemann equations (25) on a vicinity of $(x,y)=(0,0)$. Since the class of these functions clearly contains pairs u, v for which u_x and u_y attain preassigned values at the origin, (37) is satisfied. But it was proved above that this is impossible.

This contradiction proves (i). But the proof depended on Lemma (†), which therefore remains to be proved.

Proof of (†). Let $[\Gamma]$ be the interior of a Jordan curve Γ which is sufficiently smooth (say, piecewise of class C^1), and let

$$u=u(x,y),\qquad v=v(x,y) \tag{39}$$

be a pair of real-valued functions which, on $[\Gamma]$, are of class C^1, uniformly continuous and such as to satisfy the pair of partial differential equations

$$u_x-v_y=\lambda(x,y),\qquad u_y+v_x=\mu(x,y), \tag{40}$$

where λ, μ are given functions which are uniformly continuous of $[\Gamma]$ (so that they, as well as the functions (39), possess continuous boundary values on Γ). Then the assertion of (†) is equivalent to the statement that, at every point (x,y) of $[\Gamma]$, the functions λ, μ must behave in such a way that, whenever $b>0$ is small enough,

$$\lim_{\epsilon\to+0}\int_0^{2\pi}\Big(\int_\epsilon^b \rho^{-1}(\lambda\cos 2\phi+\mu\sin 2\phi)\,d\rho d\phi \text{ exists} \tag{41}$$

(as a finite limit), where the argument of both λ and μ is $(x+\rho\cos\phi,$ $y+\rho\sin\phi)$. The proof (41) proceeds as follows:

If ∂ denotes the operator $(\ )_x + i(\ )_y$, then (40) can be written in the form $\partial w = \omega$, where $w = u + iv$ and $\omega = \lambda + i\mu$ (so that w is a complex-valued function of class C^1 of the real variables x, y on the domain $[\Gamma]$, on which both w and ω are uniformly continuous). Let $z = x + iy$ and $\zeta = \xi + i\eta$, and put

$$1/(\zeta - z) = G = G(x, y) = G(x, y; \xi, \eta) \tag{42}$$

if $z \neq \zeta$. Then $\partial G = 0$ is an identity in (x, y) for fixed $\zeta \neq z$. Hence the equation $\partial w = \omega$ can be written in the form $\partial(wG) = \omega G$. Consequently, an application of Green's theorem gives

$$\int_{\Gamma+C} wGdz = i \iint_B \omega G dxdy,$$

where Γ is positively oriented, $C = C(\epsilon; \xi, \eta)$ denotes the negatively oriented circle of (small) radius ϵ about the point $\zeta = \xi + i\eta$ and $B = B(\epsilon; \xi, \eta)$ is the annular domain between Γ and C. But it is clear from (42) that

$$\int_C wGdz \to 2\pi i w(\xi, \eta) \text{ as } \epsilon \to 0.$$

Hence, by letting $\epsilon \to 0$ in the preceding relation also, it follows that

$$w(x, y) + (2\pi i)^{-1} \int_\Gamma wGdz = (2\pi)^{-1} \iint_{[\Gamma]} \omega G d\xi d\eta \tag{43}$$

holds for every point (x, y) of $[\Gamma]$. (Except for the notations, (43) is the same as formula (3) of Carleman [2], p. 473; cf. formulae (28) of Lichtenstein [10]).

It is clear from (42) that the function of (x, y) represented by the line integral on the left of (43) is regular analytic in $z = x + iy$ on $[\Gamma]$ (hence such as to annihilate the operator ∂). Since w is of class C^1 and satisfies $\partial w = \omega$ on $[\Gamma]$, it follows from (43) that w^* is of class C^1 (and satisfies $\partial w^* = \omega$) on $[\Gamma]$, if $w^* = w^*(x, y)$ denotes the expression on the right of (43).

With reference to a sufficiently small $b > 0$ (which can be kept fixed for every closed (x, y)-subset of $[\Gamma]$), let $D = D(x, y)$ denote the interior of the circle of radius b about (x, y). Then, since $w^*(x, y)$ is of class C^1, the contribution of D to the double integral $w^*(x, y)$ also is a function of class C^1. In particular, both partial derivatives f_x, f_y of the function

$$f(x, y) = \iint_D (\lambda + i\mu) G(\xi, \eta; x, y) d\xi d\eta, \tag{44}$$

where $D = D(x, y)$ and $\lambda + i\mu = \omega = \omega(\xi, \eta)$, must exist. Hence the same is true of the corresponding partial derivatives of $\mathfrak{R}f(x, y)$ and $\mathfrak{J}f(x, y)$. But if

$$\xi = x + \rho \cos \phi, \qquad \eta = y + \rho \sin \phi, \tag{45}$$

it is seen from (44) and (42) that the functions $\mathfrak{R}f(x, y)$, $-\mathfrak{J}f(x, y)$ are identical with

$$\int_0^{2\pi}\int_0^{b} \rho^{-1}(\lambda \cos \phi + \mu \sin \phi) d\rho d\phi, \qquad \int_0^{2\pi}\int_0^{b} \rho^{-1}(\lambda \sin \phi - \mu \cos \phi) d\rho d\phi \tag{46}$$

(where $\lambda = \lambda(\xi, \eta)$, $\mu = \mu(\xi, \eta)$, whilst ξ, η are given by (45)). Functions of the type

$$\int_0^{2\pi}\int_0^{b} e(\phi)\rho^{-1}\nu(x + \rho \cos \phi, y + \rho \sin \phi) d\rho d\phi,$$

where $e(\phi) = \cos \phi$ or $e(\phi) = \sin \phi$ and $\nu = \lambda$ or $\nu = \mu$, are precisely those treated by Petrini [11], pp. 128-130 (such functions are, in the main, partial derivatives of the first order of logarithmic potentials; cf. *ibid.*, p. 129). Petrini's proof ([11], pp. 131-132) of his theorem (pp. 132-133) implies that the partial derivative of $\mathfrak{R}f$ with respect to x (at the point (x, y)) will exist if and only if condition (41) is satisfied. This proves (†).

Proof of (ii*). In order to prove (ii*), it will be convenient to re-examine the proof of (ii). Consider the case (24) of (2), where g is the function of class C^1 given by (29). Suppose, if possible, that the corresponding Laplace-Beltrami equation (26) has, on a vicinity of $(x, y) = (0, 0)$, a solution $v = v(x, y)$ of class C^2 satisfying

$$v_x^2(0, 0) + v_y^2(0, 0) \neq 0. \tag{47}$$

Since (26) is the integrability condition of (25), it follows that (25) defines a function $u = u(x, y)$, unique up to an additive constant. It is clear that u is of class C^2, since v is of class C^2 and g is of class C^1. It also follows from (25) that $u_x v_y - u_y v_x = g^{-1}v_x^2 + g v_y^2$, which, in view of (47) and $g(0, 0) = 1 \neq 0$, implies that $\partial(u, v)/\partial(x, y)$ does not vanish at, hence near, $(x, y) = (0, 0)$. This contradicts the proof of (ii). Consequently, any C^2-solution of (26) on a vicinity of $(x, y) = (0, 0)$ fails to satisfy (47).

The proof of (ii), and hence the last conclusion, is based essentially on two properties of g, namely, that

$$g(0, 0) = g^{-1}(0, 0) = 1 \tag{48}$$

and that, for some $b > 0$,

$$(49) \qquad \lim_{\epsilon\to 0} \int_0^{2\pi} \int_\epsilon^b r^{-1} \cos 2\theta \; h(r\cos\theta, r\sin\theta)\, dr d\theta \text{ does not exist}$$

as a finite limit, where $h = g$. If the 1 in (29) is replaced by a $c(> 0)$, so that (48) does not hold, then, in the proof of (ii), the Poisson equation (28) becomes replaced by $c^2 v_{xx} + v_{yy} = f$. This can be reduced to a Poisson equation by a change of the independent variables $(cx, y) \to (x, y)$. The condition corresponding to (49) holds if, for example,

$$(50) \qquad \lim_{\epsilon\to 0} \int_0^{2\pi} \int_\epsilon^b r^{-1} \cos 2\theta \; h(cr\cos\theta, r\sin\theta)\, dr d\theta = -\infty.$$

In the proof of (ii*), it is sufficient to observe that (50) holds for all c near 1. In order to see this, note that the integral occurring in (50) is an absolutely convergent double integral over a region bounded by the circles $r = \epsilon$ and $r = b$, where $r^2 = x^2 + y^2$. If the inner boundary is replaced by the ellipse $c^2 x + y^2 = \epsilon^2$, the domain of integration is changed so that the area of the region added or subtracted is $|1 - c^{-1}| \pi\epsilon^2$, while the integrand in this region is $O(\epsilon^{-1})/\log \epsilon$ as $\epsilon \to 0$; cf. (27). Hence the difference between the two integrals is $O(\epsilon)/\log\epsilon = o(1)$, as $\epsilon \to 0$. Thus, in proving (50), the inner and, of course, the outer boundary can be supposed to be changed to the ellipses $c^2x^2 + y^2 = \epsilon^2$, $c^2x^2 + y^2 = b^2$, respectively. The integral to be considered is then

$$c \int_0^{2\pi} \int_\epsilon^b r^{-1}(\cos^2\theta - c^2\sin^2\theta)(\cos^2\theta + c^2\sin^2\theta)^{-2} h(r\cos\theta, r\sin\theta)\, dr d\theta,$$

or, in view of (27),

$$c \int_0^{2\pi} \Big(\int_\epsilon^b r^{-1} \log r^2 \, dr\Big) \cos^2\theta (\cos^2\theta - c^2\sin^2\theta)(\cos^2\theta + c^2\sin^2\theta)^{-2} d\theta.$$

Hence, (50) holds whenever

$$\int_0^{2\pi} \cos^2\theta(\cos^2\theta - c^2\sin^2\theta)(\cos^2\theta + c^2\sin^2\theta)^{-2} d\theta > 0.$$

This is the case if $c = 1$ and hence if c is sufficiently near 1.

The proof of (ii*) can now be completed along the lines of [13], pp. 736-737, by the usual type of "Lebesgue construction," as follows:

Let $a > 0$ be chosen in (1) so small that the function g defined by (29)

is positive (and of class C^1) on (1). Let $(x_1, y_1), (x_2, y_2), \cdots$ be a sequence of distinct points of $D_{\frac{1}{2}a}$ dense on $D_{\frac{1}{2}a}$ and let δ, η be fixed positive numbers. For $n = 1$ and $n > 1$, respectively, put $A_n = 1$ and

$$(51) \qquad A_n^{-1} = \max_{1 \leqq k < n,\ |1-c| \leqq \eta} 2^n \iint_E r^{-1} \,|\, h(x_k - x_n + cr\cos\theta, y_k - y_n + r\sin\theta) - h(x_k - x_n, y_k - y_n) \,|\, dr d\theta,$$

where $E = E(k, n, c)$ indicates the (r, θ)-set for which the argument of h is a point of D_a. Clearly, $0 < A_n < \text{const.}\ 2^{-n}$ and the series

$$(52) \qquad g^*(x, y) = \delta \sum_{n=1}^{\infty} A_n g(x - x_n, y - y_n)$$

defines a function which is positive and of class C^1 on $D_{\frac{1}{2}a}$. In addition, (52) can be differentiated term-by-term. Let $\eta > 0$ be chosen so that (50) holds for every c on the range $|\, c - 1 \,| \leqq \eta$ and let $\delta > 0$ be chosen so that $g^*(0, 0) = 1$. Then $|\, g^*(x, y) - 1 \,| \leqq \eta$ if (x, y) is on D_α for some sufficiently small $\alpha (< \frac{1}{2}a)$.

Let $c = c_k = g^*(x_k, y_k)$ if (x_k, y_k) is on D_α; so that $|\, c - 1 \,| \leqq \eta$. If g is replaced by g^* in (26), the resulting equation cannot have, in a vicinity of $(x, y) = (x_k, y_k)$, a solution $v = v(x, y)$ of class C^2 satisfying

$$v_x^2(x_k, y_k) + v_y^2(x_k, y_k) \neq 0$$

if, for some $b > 0$,

$$(53) \qquad \lim_{\epsilon \to 0} \int_0^{2\pi} \int_\epsilon^b r^{-1} \cos 2\theta \; g^*_x(x_k + cr\cos\theta, y_k + r\sin\theta) \, dr d\theta = -\infty.$$

In order to verify (53), note that if the series (52) is differentiated term-by-term with respect to x and if the result is substituted into (53), then the integral occurring in (53) can be written as the sum of k integrals and a remainder term. The latter is majorized, uniformly in ϵ, by

$$\sum_{n=k+1}^{\infty} A_n I_n \leqq \text{const.} \sum_{n=k+1}^{\infty} 2^{-n},$$

where $I_n = I_n(k)$ is the integral occurring in (51). Of the first k integrals, the first $k - 1$ tend, as $\epsilon \to 0$, to finite limits, since $(x_k, y_k) \neq (x_n, y_n)$ for $k < n$, while the k-th integral is, up to a constant positive factor, the integral occurring in (50) and tends therefore to $-\infty$, as $\epsilon \to 0$.

Hence, if (26), where g is replaced by g^*, has a solution $v = v(x, y)$ of class C^2 on a subdomain of D_α, then $v_x(x_k, y_k) = v_y(x_k, y_k) = 0$ for every (x_k, y_k) on the domain of existence of v. Since $(x_1, y_1), (x_2, y_2), \cdots$ contains a subsequence dense on D_α, it follows that $v_x \equiv v_y \equiv 0$. Thus (ii*) is proved.

Proof of (i*). The proof of (i) implies that if g in the Cauchy-Riemann equations (25) is defined by (36) and if $u = u(x, y)$, $v = v(x, y)$ is a C^1-solution of (25) in a vicinity of $(0, 0)$, then $u_x = u_y = v_x = v_y = 0$ at $(x, y) = (0, 0)$. This conclusion depended on (48) and (49), where $g = 1 + h$. If the 1 in the last relation is replaced by a $c(> 0)$, then the corresponding equations (25) can be written as a system occurring in (†) after the change of independent variables $(cx, y) \rightarrow (x, y)$. The condition corresponding to (49) is implied by (50). Thus, it is clear that the proof of (i*) can be completed by arguments analogous to those used in the proof of (ii*).

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ON THE SINGULARITIES IN NETS OF CURVES DEFINED BY DIFFERENTIAL EQUATIONS.*

By Philip Hartman and Aurel Wintner.

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1. Various questions in the differential geometry of surfaces are known to depend on a differential equation of the form

$$adx^2 + 2bdxdy + cdy^2 = 0, \tag{1}$$

where a, b, c are continuous functions of (x, y). If the discriminant $ac - b^2$ is negative at a point (x_0, y_0), then (1) is equivalent (after a rotation of the (x, y)-plane) to two non-singular differential equations of the form $dy/dx = f(x, y)$, where (x, y) is confined to a sufficiently small vicinity of (x_0, y_0). If $ac - b^2$ is negative near (x_0, y_0) and vanishes at (x_0, y_0) but a, b, c do not vanish simultaneously, then (1) is still "equivalent" to two non-singular differential equations. In the latter case, the solution curves of (1) passing through (x_0, y_0) have the same tangent, whereas there are two distinct such directions when $ac - b^2$ is negative at (x_0, y_0) also.

The first part of the present paper will be concerned with (1) in the case of an isolated singular point (x_0, y_0). By this is meant that $ac - b^2$ is negative near (x_0, y_0) but vanishes at (x_0, y_0) in such a way that both a and c, and therefore all three coefficients of (1), vanish at (x_0, y_0). The results will then be applied to the lines of curvature at an isolated umbilical point (which can be either a spherical or a flat point) of a surface (Section **11**), and also to the asymptotic lines at an isolated umbilical point (which must be a flat point) of a surface of non-positive curvature (Section **12**).

2. On a vicinity of $(x, y) = (0, 0)$, let the coefficient functions of (1) be continuous functions of the form

$$a = \alpha_1 x + \beta_1 y + f_1, \quad b = \alpha_2 x + \beta_2 y = f_2, \quad c = \alpha_3 x + \beta_3 y + f_3, \tag{2}$$

where α_k, β_k are six constants and each of the three functions $f_k = f_k(x, y)$ satisfies

$$f_k(x, y) = o(r), \qquad (k = 1, 2, 3), \tag{3}$$

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as $r \to 0$, where $r = (x^2 + y^2)^{\frac{1}{2}}$. Supose further that

$$(4) \qquad ac - b^2 \leqq 0 \text{ according as } x^2 + y^2 \geqq 0.$$

By a solution path of (1) will be meant a set of points (x, y) which can be represented as a locus $x = x(\tau)$, $y = y(\tau)$, where $x(\tau)$, $y(\tau)$ are continuously differentiable functions satisfying (1) on a τ-interval and neither $(x(\tau), y(\tau))$ nor $(dx/d\tau, dy/d\tau)$ becomes the vector $(0, 0)$ at any point of that interval. The set of the solution paths of (1) contained in a vicinity of a point $(x^0, y^0) \neq (0, 0)$ can be divided into two well-distinguishable systems, since, in a vicinity of $(x^0, y^0) \neq (0, 0)$, the differential equation (1) splits into two non-singular differential equations the solution paths of which, in view of (4), have two distinct tangents at (x^0, y^0). It follows that, despite the singularity of (1) at $(0, 0)$, *all* solution paths of (1) can be divided into two distinct systems, say S_1 and S_2.

By a solution path of (1) reaching to the origin will be meant a solution path $x = x(\tau)$, $y = y(\tau)$ defined on an interval of the form $0 \leqq \tau < \tau_0$ $(\leqq \infty)$ in such a way that

$$(5) \qquad (x(\tau), y(\tau)) \to (0, 0) \text{ as } \tau \to \tau_0.$$

If r and θ denote polar coordinates,

$$(6) \qquad r = (x^2 + y^2)^{\frac{1}{2}}, \qquad \theta = \arctan y/x,$$

then, since $(x, y) \neq (0, 0)$ on a solution path, (6) defines continuously differentiable functions $r = r(\tau) > 0$, $\theta = \theta(\tau)$ when $x = x(\tau)$, $y = y(\tau)$ is a solution path.

In terms of the six constants occurring in (2), define three linear trigonometric forms by

$$(7) \qquad L_k(\theta) = \alpha_k \cos\theta + \beta_k \sin\theta, \qquad (k = 1, 2, 3),$$

and then a quadratic trigonometric polynomial by

$$(8) \qquad Q(\theta) = L_2^2(\theta) - L_1(\theta) L_3(\theta)$$

and a cubic trigonometric polynomial by

$$(9) \qquad M(\theta) = L_1(\theta)\cos^2\theta + 2L_2(\theta)\sin\theta\cos\theta + L_3(\theta)\sin^2\theta.$$

3. The following theorem will first be proved:

(*) *On a vicinity of* (x, y), *let* $a(x, y)$, $b(x, y)$, $c(x, y)$ *be continuous functions satisfying* (2), (3), (4) *and the following three conditions:*

(7 bis) $L_1(\theta), L_2(\theta), L_3(\theta)$ have no common zero;

(8 bis) $Q(\theta) \not\equiv 0$; (9 bis) $M(\theta) \not\equiv 0$.

Then, if S_1 and S_2 are defined as in Section **2**,

(I) *each of the systems S_1, S_2 of solution paths of* (1) *contains at least one solution path satisfying* (5);

(II) *there belongs to every solution path of* (1) *satisfying* (5) *an angle θ_0 such that both*

(10) $$\theta \to \theta_0, \quad \text{where} \quad \theta = \text{arc tan } y/x,$$

and

(11) $$\phi \to \theta_0 \pmod{\pi}, \quad \text{where} \quad \phi = \text{arc tan } dy/dx,$$

hold, as $(x, y) \to (0, 0)$.

The proof of this theorem (*) will be based on the results of [3].

Remark 1. In view of (3), it is clear from (4) that

(12) $$Q(\theta) \geqq 0.$$

But if (12) is strengthened to

(13) $$Q(\theta) > 0$$

(for all θ) or, what is the same thing, if (4) is strengthened to

(14) $$ac - b^2 < -\text{const.}\ r^2 < 0, \quad \text{where} \quad x^2 + y^2 = r^2 \neq 0,$$

then (7 bis) and (9 bis) are automatically satisfied; cf. (7), (8), (9). Accordingly,

(14*) (7 bis), (8 bis), (9 bis) are implied by (14).

Remark 2. If the $o(r)$-terms (3) are neglected in (2), then the coefficients of (1) become the linear forms $\alpha_k x + \beta_k y$. Hence it is seen from (7) and (9) that assumption (9 bis) of the italicized Theorem (*) can be formulated as follows: The "linear" approximation to (1) is *not* of the form

$$\beta_1 y dx^2 - (\beta_1 x + \alpha_3 y) dx dy + \alpha_3 x dy^2 = 0.$$

On the other hand, every half-line issuing from the origin of the (x, y)-plane is readily seen to be a solution path of every differential equation of the latter form. If the analogy of the corresponding differential equations of first order

$$(a_{11}x + a_{12}y)\,dy - (a_{21}x + a_{22}y)\,dx = 0$$

is considered, every half-line issuing from the point $(0, 0)$ will be a solution path if and only if the binary matrix (a_{ik}) (which should not be the zero matrix) has a multiple characteristic number but no multiple elementary divisor. Hence, in view of a counter-example which is known in this case for the (non-linear) first degree analogue of (non-linear) differential equation (1) of second degree (cf. [3], p. 123), assumption (9 bis) of (*) seems to be indispensable for the truth of (*).

In this connection, it is worth emphasizing that the assumptions of (*) do not preclude the case in which, when the terms (3) of (2) are omitted, (1) factors into

$$dx \cdot \{(\beta x + \alpha y)\,dx - (\alpha x - \beta x)\,dy\} = 0,$$

where both constants α, β can be distinct from 0. But then the vanishing of the second factor is a linear differential equation $\{\quad\} = 0$ for which the point $(0, 0)$ is a vortex.

4. If $\theta = \theta_i$ and $\theta = \theta_i + \pi$, where $1 \leqq i \leqq h$, denote the $2h$ distinct $(\operatorname{mod} 2\pi)$, real roots of $M(\theta) = 0$ (so that $h = 1, 2$ or 3), then it will be clear from the proof below that the assertions of (*) above can be amplified as follows: The numeration of these roots can be chosen in such a way that, if S_1 and S_2 denote the systems defined above (before formula (5), in Section **2**) and if the limit θ_0 occurring in (10)-(11) is not a zero of $Q(\theta)$, then θ_0 will be $(\operatorname{mod} 2\pi)$ a root θ_i or a root $\theta_i + \pi$ according as the solution path considered in (10)-(11) belongs to S_1 or to S_2. In addition, if $\theta = \theta_0$ is a root of odd order of $M(\theta) = 0$, then one of the two systems S_1, S_2 must contain a solution path satisfying (5) and (10), while the other system must contain a solution path satisfying (5) and what results if θ_0 is replaced by $\theta_0 + \pi$ in (10). Finally, if two solution paths reaching to the origin belong to one and the same θ_0 $(\operatorname{mod} 2\pi)$ in (10), and if $Q(\theta_0) \neq 0$, then both paths are contained in one and the same S_j $(j = 1, 2)$.

Since (*) requires only the continuity of the functions (3), there is assumed no local uniqueness (say a Lipschitz condition) for the solution paths of either system S_j at points (x, y) distinct from $(0, 0)$. But this generality will not be the only new aspect in the proof below, since the literature consulted fails to contain a proof of (*) even for the case in which the functions (2) are analytic and (14) holds. In fact, the situation is as follows:

For "generic" values of the six constants in (7), the assertions of (*)

were obtained by Picard [5], p. 224 (cf. Liebmann's report [4]) under the assumption that the functions (3) are regular power series about the origin (actually, somewhat less is used *loc. cit.*). A careful perusal of Picard's considerations shows however that this proof is wrong (even if the functions (3) are polynomials). For, in order to show that in the case of (14) no solution path reaching to the origin is a spiral (i. e., that $|\theta(\tau)| \rightarrow \infty$ cannot take place as $\tau \rightarrow \tau_0$), Picard assumes (*loc. cit.*, p. 221) that one of the two systems S_j, say S_1, contains two solutions reaching to the origin in such a fashion that one of the solutions satisfies (10) for some θ_0 and the other solution satisfies the relation which results from (10) if that θ_0 is increased by π. Actually, such an assumption cannot be made; in fact, as observed above, when $Q(\theta_0) \neq 0$, it is impossible that the latter of the solution paths be in S_1, since the former is assumed to be in S_1. Incidentally, if this result is granted, the error could be eliminated by using (*via* analyticity or less) the local uniqueness of solution paths passing through any point distinct from $(0, 0)$, since then it is easy to see that no solution path reaching to the origin can be a spiral.

An objection can also be made to the passage in which Picard assumes (*loc. cit.*, p. 219) that $\alpha_1\beta_3 - \alpha_3\beta_1 \neq 0$ holds for the constants occurring in the above relations (2). This assumption is *always* violated in the principal application made by Picard (*loc. cit.*, p. 225) of his result, namely, in the case in which (1) is the equation of the lines of curvature on a surface on which $(x, y) = (0, 0)$ is an isolated umbilical point. In fact, $\alpha_1 = -\alpha_3$ and $\beta_1 = -\beta_3$ always hold in this geometrical problem; so that $\alpha_1\beta_3 - \alpha_3\beta_1 = 0$ for any choice of the coordinate axes in the (x, y)-plane.

5. Assumption (7 bis) of (*) is violated if and only if there exist a homogeneous linear trigonometric form

$$L(\theta) = \alpha\cos\theta + \beta\sin\theta \not\equiv 0 \qquad (\text{i. e., } \alpha^2 + \beta^2 \neq 0) \tag{15}$$

and three constants c_k satisfying

$$L_k(\theta) = c_k L(\theta), \qquad (k = 1, 2, 3), \tag{16}$$

which implies, by (8) and (9), that

$$Q(\theta) = (c_2{}^2 - c_1c_3)L^2(\theta) \tag{17}$$

and

$$M(\theta) = (c_1\cos^2\theta + 2c_2\cos\theta\sin\theta + c_3\sin^2\theta)L(\theta), \tag{18}$$

where, in view of (12) and (8 bis),

(18 bis) $$c_2^2 - c_1c_3 > 0.$$

We were unable to decide whether or not (*) remains true in this case, that is, when assumption (7 bis) is omitted. We shall prove however, by a method which combines that proving (*) with that of "the curves of zero slope" (cf. [8]), that *if the coefficient functions of* (1) *are of class* C^1 (*instead of being, as in* (*), *just continuous*), then, if condition (7 bis) of (*) fails to hold, the assertions of (*) remain true at least in the following sense:

(§) *Assertion* (I) *of* (*) *remains true, as does that part of assertion* (II) *which concerns* (10); *the remaining part of* (II), *that which concerns* (11), *is true at least if the limit* θ_0 *occurring in* (10) *is not a zero of* (15).

The proof of this variant of (*), being made quite elaborate by the necessity of involving the "curves of zero slope," will be deferred to Section **15**. On the other hand, it will be shown in Section **14** that, even when the C^1-assumption of the variant is omitted, statement (§), the last italicized statement, holds if (7 bis) is replaced by the assumption that all (real) zeros of (9) are simple (i. e., if $M(\theta)$ and $dM(\theta)/d\theta$ do not vanish simultaneously). In contrast to the former variant of (*), the latter variant of (*) can be proved with not more effort than (*) itself.

6. If the (x, y)-plane is rotated about $(0, 0)$ by any fixed angle, say by θ^*, the form of the equations (1)-(3) remains unchanged. In fact, if $L_k^*(\theta)$, $Q^*(\theta)$, $M^*(\theta)$ denote the trigonometric polynomials by which (7), (8), (9) become replaced after the rotation

(19) $$(x, y) \rightarrow (x \cos \theta^* - y \sin \theta^*, \qquad x \sin \theta^* + y \cos \theta^*),$$

then it is readily verified from (1)-(3) and (6)-(7) that, in vector and matrix notations,

(20_1) $$\begin{bmatrix} L_1^*(\theta) \\ L_2^*(\theta) \\ L_3^*(\theta) \end{bmatrix} = \begin{bmatrix} \cos^2\theta^* & \sin 2\theta^* & \sin^2\theta^* \\ -\frac{1}{2}\sin 2\theta^* & \cos 2\theta^* & -\frac{1}{2}\sin 2\theta^* \\ \sin^2\theta^* & -\sin 2\theta^* & \cos^2\theta^* \end{bmatrix} \begin{bmatrix} L_1(\theta+\theta^*) \\ L_2(\theta+\theta^*) \\ L_3(\theta+\theta^*) \end{bmatrix}$$

and therefore, from (8) and (9),

(20_2) $$Q^*(\theta) = Q(\theta+\theta^*); \qquad (20_3) \quad M^*(\theta) = M(\theta+\theta^*).$$

While a rotation (19) leaves the zeros of $M(\theta)$ invariant in the sense of (20_3), the positions of the zeros of the functions $L_k(\theta)$ do not remain invariant relative to the position of the zeros of $M(\theta)$; cf. (20_1). This is the reason for the possibility of the following consideration.

It is clear from (7) that, when θ is fixed, $L_k{}^*(\theta-\theta^*)$ is a quadratic form in $(\cos\theta^*, \sin\theta^*)$. If $k=3$, the coefficients of this quadratic form are seen to be $L_3(\theta)$, $-L_2(\theta)$, $L_1(\theta)$. Hence the assumption (7 bis) means that for no fixed θ will $L_3(\theta-\theta^*)$ vanish identically in θ^*. In view of (8), (8 bis) and (20_1), the function $L_3{}^*(\theta-\theta^*)$ of θ^* has, for a fixed θ, exactly one zero or exactly two zeros (mod π) according as θ is or is not a zero of the quadratic form $Q(\theta)$ in $(\cos\theta, \sin\theta)$.

Suppose that θ_0 is a zero of $M(\theta)$. Thus, (20_3) shows that $\theta=\theta_0-\theta^*$ is a zero of $M^*(\theta)$. Hence, in view of the preceding remarks on $L_3{}^*(\theta-\theta^*)$, it is seen from (9 bis) that, when θ^* is suitably chosen, $L_3{}^*(\theta)$ and $M^*(\theta)$ will not have a common zero, which means that $L_3{}^*(\theta)\neq 0$ holds at $\theta=\theta_0-\theta^*$ whenever $M(\theta_0)=0$. It also follows that if ϑ is any preassigned angle, then θ^* in (19) can be chosen in such a way that $L_3{}^*(\theta)\neq 0$ whenever $M^*(\theta+\vartheta)=0$, that is, whenever $\theta=\theta_0-\theta^*-\vartheta$ and $M(\theta_0)=0$.

Consequently, if the rotation (19) is suitably chosen and then the asterisks are omitted, it follows that there is no loss of generality in assuming that

$$L_3(\theta_0)\neq 0 \text{ if } M(\theta_0)=0 \tag{21}$$

and that, with reference to a preassigned ϑ,

$$L_3(\theta_0+\vartheta)\neq 0 \text{ if } M(\theta_0)=0. \tag{22}$$

The hypothesis of (21)-(22) is always satisfied by some θ_0. In fact, since (9) is a cubic form in $(\cos\theta, \sin\theta)$, it must have a (real) zero, say θ_0. In adidtion, (9 bis) shows that $M(\theta)$ will change sign at θ_0 if θ_0 is suitably chosen.

7. Starting with the coefficients of (1), which (for small x^2+y^2) are continuous functions satisfying (4), consider either of the binary differential systems

$$x'=c, \qquad y'=-b+(-1)^j(b^2-ac)^{\frac{1}{2}}, \tag{23_j}$$

where $j=1,2$, the prime denotes differentiation with respect to a variable t which does not occur explicitly in (23_j), and the exponent $\frac{1}{2}$ refers to the non-negative determination of the square root.

In contrast to the definition of a solution path of (1), given in Section **2**, where neither $(x(\tau), y(\tau))=(0,0)$ nor $(dx(\tau)/d\tau, dy(\tau)/d\tau)=(0,0)$ has been allowed, let a solution path $(x(t), y(t))$ of either system (23_j) be defined so as to exclude $(x(t), y(t))=(0,0)$ for every t, without excluding $(x'(t), y'(t))=(0,0)$. Clearly, (6) and every solution path of (23_j) deter-

mine two continuously differentiable functions $r = r(t) > 0$, $\theta = \theta(t)$ (with $0 \leqq \theta(t^0) < 2\pi$ at a given t^0). If a solution path of (23_j) is given for $0 \leqq t < t_0(\leqq \infty)$ or $0 \geqq t > t_0(\geqq -\infty)$ and if it satisfies

$$(24) \qquad (x(t), y(t)) \to (0, 0) \text{ as } t \to t_0,$$

then it will be called a solution path of (23_j) reaching to the origin.

The pair of alternative systems (23_1)-(23_2) of first order is *formally* equivalent to the single equation (1) of second order. But from the point of view of *solution paths*, the equivalence is not evident at all. In fact, a solution path of (1), as defined in Section **2**, depends on the idea of a (locally) Jordan arc, of class C^1, on which the parameter τ is in the main the arc length, whereas the t in (23_j), where $' = d/dt$, is committed by the assignment of the slope functions of both $z = x(t)$ and $z = y(t)$ in a (t, z)-plane. Correspondingly, when proving that every solution path of (1) satisfying (5) can be thought of as a solution path of either (23_1) or (23_2) and vice versa, one meets the actual difficulty at the points (x, y) at which $(x'(t), y'(t)) = (0, 0)$ by virtue of (23_j).

8. Let the functions F and G_1, G_2 be defined by

$$(25) \qquad F(\theta) = L_3(\theta)$$

and

$$(26_j) \qquad G_j(\theta) = -L_2(\theta) + (-1)^j Q^{\frac{1}{2}}(\theta),$$

where $Q^{\frac{1}{2}}$ denotes the non-negative square root of (8). Then it is clear from (2)-(3) and (7)-(8) that $1/r$ times the functions on the right of the equations (23_j) tend, uniformly in θ, to the limits $F(\theta)$ and $G_j(\theta)$, as $r \to 0$. The set of θ-values on which $F^2 + G_j^2$ vanishes is contained in the set of θ-values on which F vanishes, and the latter set is a sequence of the form $\theta = \theta^0 + n\pi$ $(n = 0, \pm 1, \pm, \cdots)$, by (25) (and since L_3 is a homogeneous linear trigonometric polynomial which, in view of (21), does not vanish identically).

If

$$(27_j) \qquad J_j(\theta) = G_j(\theta) \cos\theta - L_3(\theta) \sin\theta, \qquad (L_3 = F),$$

then it is seen from (9) that

$$(28) \qquad M(\theta) L_3(\theta) = J_1(\theta) J_2(\theta).$$

Since (9 bis) and the preceding parenthetical remark imply that the trigonometric polynomial ML_3 does not vanish identically, it follows from (28) that the zeros θ of neither function J_j have a finite cluster point. This proves that assumption (†) of [3], p. 118, is satisfied.

It follows that, in order to make the three Theorems (i), (iii), (iv) of [3], pp. 118-119, applicable to both of the above systems (23_j), it is sufficient to show that, whether $j = 1$ or 2, the function J_j must change sign at some $\theta = \theta_0{}^j$, while $L_3(\theta_0{}^j) \neq 0$. If the existence of $\theta_0{}^j$ is shown, then the theorems just mentioned imply that

(I_0) *the system* (23_j) *has solution paths reaching to the origin and satisfying* (10), *where* $\theta_0 = \theta_0{}^j$;

(II_0) *to every solution path of* (23_j) *reaching to the origin, there belongs a number* θ_0 *satisfying* (10) *and* $J_j(\theta_0) = 0$;

(III_0) *the limit* θ_0 *in* (10) *satisfies* (11) *whenever* $L_3(\theta_0) \neq 0$.

After the proof of the existence of $\theta_0{}^j$, there will remain to be shown that these results can be transferred from (23_j) to (1). That (I_0) above implies (I) in (*) will be verified at the beginning of Section **10**; that (II_0) and (III_0) imply (II) in (*) will be shown at the end of Section **10** on the basis of some facts to be collected in Section **9**.

In the proof for the existence of a $\theta = \theta_0{}^j$ at which $J_j(\theta)$ changes sign, recourse can be had to the identity

$$(29) \qquad J_1(\theta) = J_2(\theta + \pi),$$

which is clear from (27_j) and (26_j), where $Q^{\frac{1}{2}} \geqq 0$.

As mentioned at the end of Section **6**, there exist values θ_0 satisfying (21) and having the property that $M(\theta)$ vanishes at θ_0 in an odd order. This means that the product $M(\theta)L_3(\theta)$ must change sign at θ_0. Hence the same is true, by (28), of either $J_1(\theta)$ or $J_2(\theta)$. It follows therefore from (29) that each of the functions $J_j(\theta)$ has a zero (θ_0 or $\theta_0 + \pi$) at which it changes sign. Thus a $\theta_0{}^j$ with the desired properties exists and can be identified with θ_0 for one choice of $j = 1, 2$, and with $\theta_0 + \pi$ for the other choice. Hence (I_0), (II_0), (III_0) are applicable to the system (23_j).

9. It is clear from the proofs of Theorems (i)-(iv) in [3,] pp. 119-122, that if θ^0 (in contrast to θ_0) is any angle satisfying $J_1(\theta^0) \neq 0$, then there exists an $s > 0$ having the following property: If

$$(30) \qquad x = x(t), \quad y = y(t), \quad \text{where} \quad t_1 \leqq t \leqq t_2,$$

is any solution of (23_1) and is within the circle C_s: $x^2 + y^2 \equiv r^2 < s^2$ for every t between t_1 and t_2, then, as t increases, the arc (30) can cross the half-line

$$(31) \qquad \theta = \theta^0 \qquad (0 < r < \infty)$$

only in one and the same direction, that is, either with increasing $\theta(t)$ only or with decreasing $\theta(t)$ only, where $\theta(t) = \arctan y(t)/x(t)$. Moreover (cf. *loc. cit.*), if

$$(32) \qquad \theta^1 < \theta < \theta^2 \pmod{2\pi}, \qquad (0 < r < \infty),$$

is a wedge in the (x, y)-plane having the property that the function $J_j(\theta)$ changes sign on the interval $\theta^1 < \theta < \theta^2$, and if (30) is any solution arc of (23_1) which is within the circle C_s and enters the wedge (32) at some t-value, then it cannot leave (32) at a larger t-value. Analogous remarks apply to $J_2(\theta)$ and the solution paths of (23_2).

10. As in Section **2**, let S_1, S_2 denote the two systems of solution paths of (1). It is seen from (4) that, after a suitable numeration $(j = 1, 2)$ of these systems, it can be assumed that if $(x(\tau), y(\tau))$ is a solution path belonging to S_j and if it does not pass through a zero $(x, y) \neq (0, 0)$ of the coefficient $c = c(x, y)$ of (1), that is, if $c(x(\tau), y(\tau)) \neq 0$, then this solution path of (1) can be reparametrized into a solution path $(x(t), y(t))$ of (23_j). Conversely, if $(x(t), y(t))$ is a solution path of (23_j) and satisfies (10) with a θ_0 subject to the restriction $L_3(\theta_0) \neq 0$, then $c(x(t), y(t)) \neq 0$ holds as soon as $(x(t), y(t))$ is close enough to $(0, 0)$. In fact, if $(x, y) \rightarrow (0, 0)$ and $c(x, y) = 0$, then $\arctan y/x \rightarrow \theta^0$ where $L_3(\theta^0) = 0$. This follows from (2), (3) and the case $k = 3$ of (7) (unless $L_3(\theta) \equiv 0$, a possibility which, in view of Section **6**, can be disregarded). Finally, it is clear that if $(x(t), y(t))$ is a solution path of (23_j) satisfying $c(x(t), y(t)) \neq 0$ throughout, then it is a solution path of (1) contained in the system S_j.

In view of result (I_0) of Section **8** on the solution paths of (23_j), this proves assertion (I) of (*) for the solution paths of (1). It remains to show that assertion (II) of (*) can be deduced from the analogous statements (II_0), (III_0).

Consider a solution path $(x(\tau), y(\tau))$ of (1) which belongs, for example, to S_1 and tends, as $\tau \rightarrow \tau_0 - 0$, to the origin $(0, 0)$ in such a way that $c = c(x(\tau), y(\tau))$ becomes 0 for certain τ-values arbitrarily close to τ_0. Let θ^0 be an angle $(\mathrm{mod}\, \pi)$ satisfying $L_3(\theta^0) = 0$ and having the property that the function $J_1(\theta)$ changes sign at some point of the interval $\theta^0 < \theta < \theta^0 + \pi$, say at the point $\theta = \theta_0$. Finally, let ϵ be any (sufficiently small) positive number satisfying $J_1(\theta^0 \pm \epsilon) \neq 0$ and $J_1(\theta^0 + \pi \pm \epsilon) \neq 0$. Then it follows from Section **9** that, when τ is close enough to τ_0, either every or no point of

the path $(x(\tau), y(\tau))$ is within the wedge (32) belonging to $\theta^1 = \theta^0 + \epsilon$, $\theta^2 = \theta^0 + \pi - \epsilon$. But it is clear from (1) and (4) that the first of these two cases is impossible, since $c(x(\tau), y(\tau))$ is supposed to become 0 at certain τ-values arbitrarily close to τ_0. Consequently, as $\tau \to \tau_0$,

$$(33) \qquad \theta^0 + \epsilon < \theta(\tau) < \theta^0 + \pi - \epsilon \text{ does not hold for any } \tau,$$

where $x(\tau) = r(\tau)\cos\theta(\tau), y(\tau) = r(\tau)\sin\theta(\tau)$; cf. (5)-(6).

Since $\epsilon > 0$ can be chosen arbitrarily small, the boundaries of the wedge prohibited by (33) can be made to be arbitrarily close to the half-lines $\theta = \theta^0 - \epsilon$ and $\theta = \theta^0 + \pi + \epsilon$. Consider, for instance, the former half-line. Then, if the path $(x(\tau), y(\tau))$ crosses at all this half-line (belonging to $\theta^0 - \epsilon$) at a certain $\tau = \tau^0 (< \tau_0)$ close to τ_0, then it follows from Section **9** that the path cannot recross the half-line $\theta = \theta^0 - \epsilon$ at any later $\tau (> \tau^0)$ unless the solution path $(x(\tau), y(\tau))$ contained in S_1 cannot be reparametrized into a solution path $(x(t), y(t))$ of (23_1); that is, unless both functions c, $-b-(b^2-ac)^{\frac{1}{2}}$ of (x, y) vanish at some point of the path $(x(\tau), y(\tau))$, where $\tau^0 < \tau < \tau_0$. Hence, if $(x(\tau), y(\tau))$ crosses the line $\theta = \theta^0 - \epsilon$ at a $\tau = \tau^0$ close to τ_0 in such a way that $\theta(\tau)$ is decreasing, then it cannot recross the line at a later $\tau (> \tau^0)$ before it has crossed the half-line $\theta = \theta^0 + \pi + \epsilon$. In this argument, the half-lines $\theta = \theta^0 - \epsilon$ and $\theta^0 + \pi + \epsilon$ can be interchanged.

It follows that the solution path $(x(\tau), y(\tau))$ of S_1 either satisfies $c(x(\tau), y(\tau)) \neq 0$ for τ near τ_0 (hence is a solution of (23_1)) or one of the following three contingencies must take place:

$$(34') \qquad \theta(\tau) \to \theta^0, \qquad\qquad (34'') \qquad \theta(\tau) \to \theta^0 + \pi,$$

$$(35) \qquad \theta^0 = \liminf \theta(\tau) < \limsup \theta(\tau) = \theta^0 + \pi,$$

where $\tau \to \tau_0$.

In the first case, (II_0) of Section **8** implies that there exists a number θ_0 satisfying (10) and $J_1(\theta_0) = 0$. At this stage of the proof, it is conceivable that the limit θ_0 is the number θ^0 or $\theta^0 + \pi$, so that (34′) or (34″) holds. It will however be shown that, for a solution path $(x(\tau), y(\tau))$ of S_1 reaching to the origin, none of the possibilities (34′), (34″), (35) can hold. The elimination of (35) implies, therefore, that (10) holds. The elimination of (34′) and (34″) implies that the limit θ_0 in (10) does not satisfy $L_3(\theta_0) = 0$. Consequently, (III_0) in Section **8** shows that (11) is a consequence of (10). Hence, the proof of (*) will be complete if it is shown that each of the three contingencies (34′), (34″) and (35) leads to a contradiction.

Ad (34′)-(34″). The angle θ^0, introduced before formula (33) above, was chosen so as to satisfy the condition $L_3(\theta^0) = 0$ and therefore, in view of

(7), the condition $L_3(\theta^0+\pi)=0$ as well, and (34′) or (34″) means that (10) is satisfied (by $\theta_0=\theta^0$ or $\theta_0=\theta^0+\pi$). Let θ_0 be a zero of $M(\theta)$ and let $\vartheta=\theta^0-\theta_0$ or $\vartheta=\theta^0+\pi-\theta_0$ according as (34′) or (34″) holds. Then, after a suitable rotation (19), it can be supposed that (22) holds; thus

$$L_3(\theta_0+\vartheta)\neq 0 \quad\text{and}\quad M(\theta_0+\vartheta)\neq 0.$$

Furthermore, since the limit of $\theta(\tau)$ is invariant under rotations, $\theta(\tau)\rightarrow\theta_0+\vartheta$ as $\tau\rightarrow\tau_0$. Since $\theta\rightarrow\theta_0+\vartheta$ and $L_3(\theta_0+\vartheta)\neq 0$, it follows that $c(x(\tau),y(\tau))\neq 0$ for τ sufficiently near τ_0. Hence $(x(\tau),y(\tau))$ can be reparametrized as a solution of (23_1). But then (II_0) in Section **8** is applicable and claims that $\theta_0+\vartheta$ must be a zero of the function $J_1(\theta)$. It follows therefore from (28) that $M(\theta_0+\vartheta)L_3(\theta_0+\vartheta)=0$. Hence, the last formula line contains a contradiction.

Ad (35). This contingency can be ruled out in the same way as (34′)-(34″) above. In fact, the angular distance (mod 2π) between a zero of $L_3(\theta)$ and a zero of $M(\theta)$ is not, whereas the corresponding distance between the

$$\theta^0=\liminf\theta(\tau)=\liminf\operatorname{arc\,tan} y(\tau)/x(\tau)$$

of (35) and a zero of $M(\theta)$ is, invariant under a rotation (19) of the (x,y)-plane.

11. *Application of* (*) *to lines of curvature.* Let $z=z(x,y)$ be a function of class C^2 in a vicinity of $(x,y)=(0,0)$, and let $K=K(x,y)$ denote the Gaussian, and $H=H(x,y)$ the mean, curvature on the surface $z=z(x,y)$. Then, as is well-known, $H^2\leqq K$, where the sign of equality is characteristic of points (x,y) which are umbilical ("spherical" or "flat" according as $K>0$ or $K=0$, while $K<0$ is precluded by $H^2=K$). It will be assumed that

$$(36)\qquad H^2(x,y)\gtreqq K(x,y) \text{ according as } x^2+y^2\gtreqq 0$$

(which means that (0, 0) is an isolated umbilical point) and that

$$(37)\qquad K(0,0)\gtreqq 0$$

(so that (0, 0) can be either a spherical or a flat point). Note that (36) and (37) are compatible with the case $K(x,y)<0$, where $(x,y)\neq(0,0)$ (in which case (0, 0) must be a flat point), and not only with the usual case $K(x,y)>0$ (in which case (0, 0) is a spherical point or a flat point, depending on the alternative in (37)).

If $p=p(x,y),\cdots,t=t(x,y)$ denote the five partial derivatives

$z_x, \cdots, z_{yy}$, the differential equations defining the lines of curvature are defined by the case

$$(38)\qquad a = pqr - (1+p^2)s, \qquad 2b = (1+q^2)r - (1+p^2)t, \qquad c = (1+q^2)s - pqt$$

of (1). Condition (36) means that the resulting quadratic differential equation (1) can or cannot be reduced to non-singular differential equations in a vicinity of a point according as the latter is not or is the point $(0,0)$.

Suppose that $z = z(x,y)$, instead of being just of class C^2, is of class C^3 (as will be seen in a moment, somewhat less would suffice at the point $(0,0)$, without any additional refinement of the C^2-condition at the other points). Then, if the plane tangent to the surface at $(0,0)$ is chosen to be the (x,y)-plane, and if the orientation of the z-axis is suitably chosen, it follows from (36) and (37) that, as $r = (x^2+y^2)^{\frac{1}{2}} \to 0$,

$$(39)\qquad z(x,y) = \tfrac{1}{2}K(0,0)(x^2+y^2) + \phi(x,y)/6 + o(r^3),$$

where, if α, β, γ, δ denote the partial derivatives of third order of $z(x,y)$ at $(0,0)$, the second term on the right is defined by the cubic form

$$(40)\qquad \phi(x,y) = \alpha x^3 + 3\beta x^2 y + 3\gamma x y^2 + \delta y^3.$$

The C^3-assumption also implies that the partial derivatives of first, second and third order, for (x,y) near $(0,0)$, can be obtained by formal differentiations of the Taylor relation (40) (with $o(r^3)$ replaced by $o(r^2)$, $o(r)$ and $o(1)$, respectively). If this is substituted into (40), it is seen that (1) will satisfy conditions (2) and (3) of (*), the values of the six constants α_k, β_k in (2) being those for which the three forms (7) become

$$(41)\qquad L_3 = \beta\cos\theta + \gamma\sin\theta = -L_1, \qquad L_2 = \tfrac{1}{2}(\alpha-\gamma)\cos\theta - \tfrac{1}{2}(\beta-\delta)\sin\theta.$$

Since the Gaussian and mean curvatures are

$$(42)\qquad K = (rt - s^2)/d^4, \text{ where } d = (1+p^2+q^2)^{\frac{1}{2}},$$

and

$$(43)\qquad H = I/d^3, \text{ where } I = \tfrac{1}{2}(1+p^2)t - pqs + \tfrac{1}{2}(1+q^2)r,$$

it is easily verified from (38) that assumption (4) of Theorem (I) is now equivalent to (36). Hence, in order to render (*) applicable, only its assumptions (7 bis), (8 bis), (9 bis) remain to be assured. But if (41) is inserted into (9), it is seen that (9 bis) is satisfied if and only if

$$(44)\qquad \phi(x,y) \not\equiv 0$$

holds for the cubic form (40). On the other hand, (8) shows that (8 bis) is satisfied if and only if not both linear forms (41) vanish identically, a condition which, in view of (40), is readily found to be equivalent to (44). Finally, condition (7 bis) requires the linear independence of the two linear forms (41), and this condition is satisfied if and only if

$$(45) \qquad \alpha\gamma - \beta\delta \neq (\gamma + \beta)(\gamma - \beta).$$

Hence the situation is as follows:

Let $z = z(x, y)$ be a surface of class C^3 satisfying (36) and suppose that, when the surface is written in the form (39), the associated cubic form (40) satisfies (44) and (45). Then Theorem (*) of Section **3** is applicable to the lines of curvature near the umbilical point $(0, 0)$.

Accordingly, every line of curvature which reaches to the umbilical point $(0, 0)$ has there a tangent (and the latter is the limit, as $(x, y) \rightarrow (0, 0)$, of the tangents at the non-umbilical points (x, y) of that line of curvature). If S_1 and S_2 are the two families of lines of curvature in a vicinity of the umbilical point (cf. the remark which precedes (5) in Section **2**), then both S_1 and S_2 contain at least one curve reaching to $(0, 0)$. (Note that the curves contained in either family, say in S_1, are transversal to those contained in S_2, if the umbilical point is excluded.) Finally, if C is a line of curvature which has at $(0, 0)$ a continuous tangent and passes *through* the point $(0, 0)$ (instead of just reaching it), then *the two arcs into which* $(0, 0)$ *divides* C *cannot be in one and the same family* S_j.

All of this is in agreement with (but is not of course contained in) the particular results derived by Darboux [1], pp. 448-465, as illustrated, in part, by his diagrams, p. 455; cf. also [2], pp. 84-93.

Remark. It may be mentioned that (45) admits of a simple interpretation, as follows: While (36) requires that $H^2 - K$ should tend to 0 as $(x, y) \rightarrow (0, 0)$, the meaning of (45) is that this limit process should not take place with an exceptional rapidity, but in such a way, for some positive constant,

$$(46) \qquad H^2(x, y) - K(x, y) \geqq \text{const.}\, r^2 \text{ as } r = (x^2 + y)^{\frac{1}{2}} \rightarrow 0.$$

In fact, it is seen from (38)-(39) and (42)-(43) that (46) is equivalent to the refinement (13) of (12). But (8) shows that (13) will be satisfied by the two forms (41) if and only if the latter are linearly independent, a restriction which is equivalent to (45).

Incidentally, since the functions (38) are of class C^1 when the surface is of class C^3, the restriction (46) becomes superfluous if use is made of the

C^1-criterion mentioned in Section **5**. In fact, (46) is equivalent to (45), whereas (45) was seen to be equivalent to (7 bis) in the present case.

12. *Application of* (*) *to asymptotic lines.* The assumptions (36)-(37) mean that (0, 0) is an isolated umbilical point. If the problem of lines of curvature is replaced by that of the asymptotic lines, the resulting dual situation is as follows: (0, 0) is an isolated flat point and the surface is of negative Gaussian curvature near (0, 0). This means that

$$(47)\qquad K(x,y) \leqq 0 \text{ according as } x^2 + y^2 \geqq 0$$

and that

$$(48)\qquad H_0 = K_0 = 0, \text{ i.e., } r_0 = s_0 = t_0 = 0$$

$(r = z_{xx}, \cdots)$, where f_0 denotes the value of $f(x,y)$ at $(x,y) = (0,0)$. Clearly, these two conditions mean that, if $(x,y) \neq (0,0)$, there exist on the surface $z = z(x,y)$ (of class C^2) two distinct asymptotic directions, and that the latter become indeterminate (instead of uniting in a single determinate direction) at $(x,y) = (0,0)$.

Let the plane tangent to the surface at $(0,0)$ be chosen to be the (x,y)-plane. Then $z_0 = p_0 = q_0 = 0$, hence the second formulation in (21 bis) shows that $z(x,y) = o(r^2)$ as $(x^2 + y^2)^{\frac{1}{2}} = r \to 0$ (Taylor). Hence, if $z(x,y)$ is of class C^3, and if $\phi(x,y)$ denotes the cubic form the coefficients of which are the same as in (39), then

$$(49)\qquad z(x,y) = \phi(x,y) + o(r^3),$$

and the remark made after (39), concerning the formal differentiability of the o-term, holds for (49) also. Since the differential equations defining the asymptotic lines on the surfaces $z = z(x,y)$ result by choosing

$$(50)\qquad a = r, \qquad b = s, \qquad c = t$$

in (1), it follows that, when the surface is of class C^3, conditions (2) and (3) of Theorem (*) are satisfied, the six constants occurring in (2) being given by

$$(51)\qquad (\alpha_1, \beta_1; \alpha_2, \beta_2; \alpha_3, \gamma_3) = (\alpha, \beta; \beta, \gamma; \gamma, \delta),$$

where α, β, γ, δ are the coefficients of the cubic (40) occurring in (49). It is also seen from (50) that, in view of (42) and (47), condition (4) is satisfied. Hence, Theorem (*) will be applicable if the four constants occurring on the right of (51) are subject to the conditions required by (7 bis), (8 bis), (9 bis).

First, it follows from (7), (51) and (8) that condition (8 bis) is equivalent to the non-identical vanishing of the Hessian of the cubic form (40), that is, to the restriction

$$(52) \qquad \phi_{xx}\phi_{yy} - \phi_{xy}{}^2 \not\equiv 0, \text{ i. e., } \phi \not\equiv \lambda^3,$$

where $\lambda = \lambda(x, y)$ denotes an arbitrary linear form (the equivalence of the two assumptions (52) on the cubic form $\phi = \phi(x, y)$ is Hesse's lemma on binary forms ϕ). On the other hand, since (7), (51), (9) and (40) imply that $M(\theta) = \phi(\cos\theta, \sin\theta)$, condition (9 bis) is equivalent to (44) and is therefore implied by the preceding condition (52). Finally, this condition takes care of (7 bis) also. In fact, (40) shows that (52) can be written in the form

$$(\beta^2 - \alpha\gamma)x^2 + (\beta\gamma - \alpha\delta)xy + (\gamma^2 - \beta\delta)y^2 \not\equiv 0,$$

and is therefore equivalent to the non-vanishing of at least one of the determinants

$$\alpha\gamma - \beta^2, \qquad \alpha\delta - \beta\gamma, \qquad \beta\delta - \gamma^2,$$

a condition which, in view of (7) and (51), is identical with the restriction (7 bis).

13. In order to deal with the statements of Section **5**, concerning the case in which

(53) $\qquad L_1(\theta), L_2(\theta), L_3(\theta)$ have a common zero,

it is convenient first to deduce a normalization which belongs to (53) in the same way as the normalization obtained in Section **7** belongs to (7 bis).

Since (53) is the negation of (7 bis), the relations (15)-(18 bis) apply. Let $P(\theta)$ denote the quadratic factor on the right of (18). Then, since $M = PL$, it is seen from (15), (18) and (18 bis) that $M = 0$ must have simple (real) roots only; so that, since L is linear, P must possess at least one simple zero which is not a zero of L. Hence it is seen from the identity (20_3), and from the corresponding identity

$$(54) \qquad L^*(\theta) = L(\theta - \theta^*),$$

that $M^*(\theta)$ possess at least one simple zero which is not a zero of $L^*(\theta)$.

It is readily verified from (20_1) and (16) that $L_1{}^*(\theta)$, $L_2{}^*(\theta)$, $L_3{}^*(\theta)$ are respectively identical with $P(\theta^*)$, $\frac{1}{2}dP(\theta^*)/d\theta^*$, $P(\theta^* + \frac{1}{2}\pi)$ times $L(\theta + \theta^*)$. This, when combined with the preceding remarks, implies that, if the constant θ^* occurring in (19) is suitably chosen, then none of the

three functions $L_k^*(\theta+\theta^*)$ of θ will vanish identically and the product $L_1^*(\theta)L_3^*(\theta)$ will be non-positive throughout. (Note that if the zeros of the quadratic trigonometric polynomial $P(\theta)$ differ from each other by multiples of $\frac{1}{2}\pi$ only, then $L_1^*(\theta)L_3^*(\theta)$ cannot become non-negative unless $L_3^*(\theta)$ vanishes identically.)

Let the constant θ^* defining the rotation (19) be suitably chosen and then all asterisks omitted (so that $\theta^*=0$). Then the above remarks can be summarized as follows:

If (53) holds instead of (7 bis), then there is no loss of generality in assuming that

$$(55) \qquad L_k(\theta) \not\equiv 0, \text{ where } k=1,2,3,$$

that

$$(56) \qquad L_1(\theta)L_3(\theta) \leqq 0 \text{ for all } \theta,$$

finally that there exists a θ_0 satisfying

$$(57) \qquad L_3(\theta_0) \neq 0 \text{ and } M(\theta_0)=0 \text{ but } M_\theta(\theta_0) \neq 0,$$

where $M_\theta(\theta)=dM(\theta)/d\theta$.

14. It is now easy to prove the statement made at the end of Section **5**. The statement deals with the case in which (7 bis) is replaced by the assumption that all (real) roots of (9) are simple (i. e., that

$$(58) \qquad M_\theta(\theta) \neq 0 \text{ whenever } M(\theta)=0,$$

where $M_\theta = dM/d\theta$), and runs as follows:

(†) *If assumption* (7 bis) *of* (*) *is replaced by* (58), *then the assertions of* (*) *are true at least in the sense of* (§), *Section* **5**.

Proof. Since (57) means that $M(\theta)$ changes signs at $\theta=\theta_0$, while $L(\theta_0)\neq 0$, a major part of the proof of (*) remains applicable. In fact, the proof of assertion (I) remains valid, as does that portion of the proof of (II) according to which either the limit (10) exists or (35) holds, where $L(\theta^0)=0$ (the first of these alternative cases includes the contingencies (34′), (34″) in the present situation). Also, (10) implies (11) if the limit θ_0 in (10) is not a zero of $L(\theta)$. Thus it only remains to show that (53) and (50) exclude the possibility of (35).

According to the normalization (55), no c_k can vanish in (15). Since the equations (1), (2), $\cdots$, being homogeneous, can be multiplied by an arbitrary non-vanishing constant, it follows that it can be supposed that $c_3=1$,

that is, that $L_3(\theta)=L(\theta)$. Then (25) reduces to

$$F(\theta)=L(\theta) \tag{59}$$

and (26) to

$$G_j(\theta)=L(\theta)(-c_2+(-1)^j(c_2{}^2-c_1c_3)^{\frac{1}{2}}\operatorname{sgn}L(\theta)), \tag{60}$$

by (17). Put

$$N_j(\theta)=(-c_2+(-1)^j(c_2{}^2-c_1c_3)^{\frac{1}{2}})\cos\theta-c_3\sin\theta. \tag{61}$$

Then the definition (27_j) of $J_j(\theta)$ shows that

$$J_j(\theta)=L(\theta)N_i(\theta) \tag{62}$$

holds in the half-plane $L(\theta)\geqq 0$ or $L(\theta)\leqq 0$ according as $j=i$ or $j\neq i$. Since $c_3=1$, it is readily verified that $P=N_1N_2$, hence $M=LN_1N_2$. It follows therefore from (58) that the zeros of N_j are distinct from those of L. Consequently, (62) shows that $J_j(\theta)$ changes sign in both of the half-planes

$$\theta^0<\theta<\theta^0+\pi, \qquad \theta^0+\pi<\theta<\theta^0+2\pi$$

if $L(\theta^0)=0$.

Accordingly, if $(x(t),y(t))$ is a solution path of (23_j) on some t-interval, is within the circle $x^2+y^2<s^2$ on this interval (for a sufficiently small s) and enters one of the wedges

$$\theta^0+\epsilon<\theta<\theta^0+\pi-\epsilon, \qquad \theta^0+\pi+\epsilon<\theta<\theta^0+2\pi-\epsilon,$$

then it cannot leave that wedge; cf. the remarks concerning (32). This fact eliminates the possibility of (35), since a solution path of (1) in such a wedge is a solution path of (23_j) (either for $j=1$ or for $j=2$), and conversely.

15. There will now be proved the C^1-criterion announced in Section **5**, that is, the following theorem:

(**) *If assumption* (7 bis) *of* (*) *is omitted but the coefficient functions a, b, c of* (1) (*which in* (*) *are required to be just continuous*) *are assumed to have continuous partial derivatives* $a_x,\cdots,c_y$, *then the assertions of* (*) *remain true at least in the sense of* (§), Section **5**.

Proof. In view of (*) and (†), it will be sufficient to prove this theorem only for the case excluded by (*) and (†) together, that is, for the case in which neither the assumption (7 bis) of (*) nor assumption (58) of (†) is

fulfilled. Since there exists a $\theta = \theta_0$ satisfying (57) in this case also, the proofs of (*) and (†) show that (**) will be proved if contingency (35) is eliminated for every solution path of (1) reaching to the origin.

The considerations of Section **13** make it clear that there is no loss of generality in assuming that $\theta^0 \neq \frac{1}{2}\pi \pmod{\pi}$ if $L(\theta^0) = 0$. This means that the coefficient β of $\sin\theta$ in (15) is not 0. Since $f_k(x, y)$ in (2)-(3) is supposed to be of class C^1, it follows that the partial derivative of a, b, c with respect to y at $(0, 0)$ is $c_1\beta$, $c_2\beta$, $c_3\beta$, respectively. Since $c_k \neq 0$ and $\beta \neq 0$, continuity considerations show that a_y, b_y, c_y are distinct from 0 for (x, y) near $(0, 0)$. Hence, the k-th of the equations

$$(63_1)\quad a(x, y) = 0; \qquad (63_2)\quad b(x, y) = 0; \qquad (63_3)\quad c(x, y) = 0$$

has a unique solution $y = y(x)$ of class C^1 for small $|x|$, say the solution

$$(64_k)\qquad y = y_k(x), \text{ where } y_k(0) = 0 \text{ and } dy_k(0)/dx = \tan\theta^0.$$

Since (4) implies that a point $(x, y) \neq (0, 0)$ of the curve (63_2) cannot be on either of the curves (63_1), (63_3), and since (63_k) is equivalent to (64_k) for every fixed k, it is clear that

$$(65)\qquad \text{either } y_2(x) > y_1(x) \text{ or } y_2(x) < y_1(x) \text{ for all small } x > 0$$

and that

$$(66)\qquad \text{either } y_2(x) > y_3(x) \text{ or } y_2(x) < y_3(x) \text{ for all small } x > 0,$$

finally that such alternatives hold for all small $x < 0$ also.

On a τ-interval $0 \leqq \tau < \tau_0$, let

$$(67)\qquad x = x(\tau), \qquad y = y(\tau)$$

be a solution path of (1) satisfying (5) as $\tau \to \tau_0$, and let (67) belong, for example, to the family S_1. On a sufficiently small τ-vicinity of any fixed τ, the coordinates of the solution path (67) must satisfy at least one of the differential equations

$$(68)\qquad dy/dx = \{-b - (b^2 - ac)^{\frac{1}{2}}\}/c, \qquad dx/dy = c/\{-b - (b^2 - ac)^{\frac{1}{2}}\}$$

(where $\{\ \}/c$ is meant to represent $-\frac{1}{2}a/b$ if $c = 0$ and $b < 0$, hence $-b - (b^2 - ac)^{\frac{1}{2}} = 0$), as well as at least one of the differential equations

$$(69)\qquad dx/dy = \{-b + (b^2 - ac)^{\frac{1}{2}}\}/a, \qquad dy/dx = a/\{-b + (b^2 - ac)^{\frac{1}{2}}\}$$

(with an analogous interpretation of $\{\ \}/a$ if $a = 0$ and $b > 0$). It is clear from these differential equations that, at a given τ, the first of the

function (67) has a relative extremum (maximum or minimum) if and only if

(70) $\qquad c(x(\tau), y(\tau))$ changes signs and $b(x(\tau), y(\tau)) > 0$

at that τ, and that the second of the functions (67) has a relative extremum if and only if

(71) $\qquad a(x(\tau), y(\tau))$ changes signs and $b(x(\tau), y(\tau)) < 0$.

On the basis of these facts, the possibility of (35) can be ruled out as follows:

Suppose, if possible, that (35) is satisfied by the solution path (67) of (1) reaching to the origin, as $\tau \to \tau_0 - 0$. Then, corresponding to every $\epsilon > 0$, and for every τ close enough to τ_0, the path (67) is in the wedge

(72) $$\theta^0 - \epsilon < \theta < \theta^0 + \pi + \epsilon \qquad (r > 0),$$

and (67) crosses the half-line

(73) $$\theta = \theta^0 + \epsilon \qquad (r > 0)$$

an infinity of times as $\tau \to \tau_0$. On the half-line (73) (for small r), the slope dy/dx in (68) and/or (69) is of constant sign. It follows that each of the functions (67) has an infinity of extrema as $\tau \to \tau_0$. These extremal values must be attained when $(x(\tau), y(\tau))$ is in one of the wedges $|\theta - \theta^0| < \epsilon$, $|\theta - \theta^0 - \pi| < \epsilon$, since these wedges contain the arcs (64_k). This follows from the above criteria involving (70) and (71).

The normalization (55) and (2)-(3) imply that, if s is sufficiently small, the sector

(74) $$\theta^0 - \epsilon < \theta < \theta^0 + \epsilon, \qquad 0 < r < s$$

is divided by the arc (64_k) into two domains on which $a_k > 0$ and $a_k < 0$, respectively, where $a_1 = a$, $a_2 = b$, $a_3 = c$. Since both (70) and (71) occur infinitely often as $\tau \to \tau_0$, it follows that the arc (63_1) is in the set $b < 0$ and that the arc (63_3) is in the set $b > 0$; cf. (65) and (66). Hence, exactly one of the functions a, c has opposite signs on the half-line (73) and on the arc (63_2). (For example, if $b > 0$ on (73) for small $r > 0$, then c has opposite signs on (73) and on $b = 0$, while a is of the same sign.)

According to (4), $a(x,y)c(x,y) < 0$ if $b(x,y) = 0$ and $(x,y) \neq (0,0)$. On the other hand, the normalization (56) implies that

$$L_1(\theta^0 + \epsilon) L_3(\theta^0 + \epsilon) < 0;$$

hence, by (2)-(3), $a(x,y)c(x,y) < 0$ on the half-line (73) for small $r > 0$. Clearly, this contradicts the fact that exactly one of the functions a, c has opposite signs on (63_2) and on (73).

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ON THE THIRD FUNDAMENTAL FORM OF A SURFACE.*

By PHILIP HARTMAN and AUREL WINTNER.

Part I.

1. Let D be an open, simply connected (and, for the purposes at hand, sufficiently small) domain in a (u, v)-plane, and let $g_{11}, g_{12} = g_{21}, g_{22}$ be three functions on D corresponding to which the quadratic form

$$(1) \qquad g_{\alpha\beta}(u^1, u^2)\,du^\alpha du^\beta, \qquad \text{where } u^1 = u, u^2 = v,$$

is positive definite at every point of D. Suppose further that the three functions $g_{ik}(u, v)$ are of class C' (i. e., that the partial derivatives Γ_u, Γ_v of the "vector"

$$(2) \qquad \Gamma = (g_{11}, g_{12}, g_{22})$$

exist and are continuous on D). Then (1) will be referred to as a *C'-metric (on D)*. Since the second derivatives of (2) need not exist, (1) will not in general have a Gaussian curvature $K = K(u, v)$. On the other hand, since

$$(3) \qquad g \neq 0, \quad \text{where } g = (\det g_{ik})^{\frac{1}{2}},$$

there exist on D continuous Christoffel coefficients

$$(4) \qquad \Gamma^j_{ik} = \Gamma^j_{ik}(u, v) = \Gamma^j_{ki}.$$

This implies that Levi-Civita's parallel transport is uniquely defined along every oriented C'-arc (contained in D) and that there exists a continuous geodesic curvature $\kappa = \kappa(s)$ on every C''-arc.

If

$$(5) \qquad u^* = u^*(u, v), \qquad v^* = v^*(u, v)$$

is a pair of functions which is of class C' and of non-vanishing Jacobian on D, and if D^* denotes the (u, v)-domain on which the (sufficiently small) (u, v)-domain D is mapped by (5), finally if

$$(6) \qquad g^*_{\alpha\beta}(u^{*1}, u^{*2})\,du^{*\alpha} du^{*\beta}, \quad \text{where } u^{*1} = u^*, u^{*2} = v^*,$$

* Received October 13, 1952.

is the positive definite form which is identical with the C'-metric (1) by virtue of (5), then (6) need not be a C'-metric on D^*. In fact, if the transformation (5) and its inverse are just of class C', then the functions $g^*_{ik}(u^*, v^*)$, instead of being of class C', will be just continuous on D^*. All that follows is that if (1) is a C'-metric on D, then (6) will be a C'-metric on D^* when the transformation (5) is of class C'' and of non-vanishing Jacobian. Conversely, it was shown in [5] that the transformation (5) must be of class C'' if it is of class C', has a non-vanishing Jacobian and transforms a C'-metric (1) on D into a C'-metric (6) on D^*.

As mentioned above, the Gaussian curvature $K = K(u, v)$ of a C'-metric does not in general exist. It is possible however to define with reference to every C'-metric *the total curvature*

$$\tau = \tau(E) \tag{7}$$

of certain subsets E of D, as follows: If $E = E(J)$ is the interior of an oriented Jordan curve J contained in D and consisting of a finite number of arcs each of which is of class C', let the total curvature (7) of E be defined as the oriented variation (with respect to the tangent vector) of the direction of a vector transported parallel to itself (Levi-Civita) along $J = J(E)$. Clearly, this definition of the set function τ remains invariant under one-to-one C''-transformations (5) which transform the given C'-metric (1) into a C'-metric (6).

If the continuous functions (4), occurring in the definition of (7), are expressed in terms of the partial derivatives $g_{ik\,u}$, $g_{ik\,v}$, it is readily found that

$$\tau(E) = \int_J (2g_{11}g)^{-1}\{(g_{12}g_{11\,u} + g_{11}g_{11\,v} - 2g_{11}g_{12\,u})du + (g_{12}g_{11\,v} - g_{11}g_{22\,u})dv\} \tag{8}$$

(cf. [1], pp. 123; this corrects the explicit form of the integrands in formulae (3)-(6) of [10], pp. 877-878). A partial integration (Green) transforms the explicit representation (8) of (7) into

$$\tau(E) = -\iint_E (4g^3)^{-1}\det(\Gamma, \Gamma_u, \Gamma_v)\,du\,dv + \int_J (2g)^{-1}\{(g_{11\,v} - g_{12\,u})du + (g_{12\,v} - g_{22\,u})dv\}, \tag{9}$$

where g^3 denotes the cube of (3) and Γ_u, Γ_v are the partial derivatives of (2).

Without using these explicit representations of (7), the following definition will now be introduced: Let a C'-metric (1) on D be called a *Gaussian metric* or such as to possess a *curvature* $K = K(u, v)$ if the set function (7), representing the total curvature of an arbitrary $E = E(J)$, is absolutely continuous, that is, if there exists on D a point function $K = K(u, v)$ which is integrable (L) on every compact subset of D and is such that, if $g = g(u, v)$ is defined by (3), then

$$\tau(E) = \iint_E Kg\,du\,dv \tag{10}$$

holds for every $E = E(J)$. This definition of the curvature, initiated by Weyl ([18], pp. 42-44; cf. [13], p. 135) leaves K (if it exists) undetermined on sets of measure 0. It is however clear what will be meant by a C'-metric which possesses a bounded curvature $K(u, v)$, a continuous curvature $K(u, v)$, etc.

Under the assumption that the C'-metric (1) is *Gaussian*, let (8*), (9*) denote the relations which result if (10) is substituted into (8), (9), respectively. Then the above introduction of a *curvature* K is justified by the following pair of facts (neither of which is contained in the other): On the one hand, if the coefficient functions of (1) are of class C'', so that the classical formula for the Gaussian curvature defines a (continuous) function $K = K(u, v)$, and if the latter is substituted into (10), then the resulting set function (10) satisfies (8*). On the other hand, if (1) is the C'-metric on a surface $X = X(u, v)$ of class C'', where $X = (x, y, z)$, and if the (continuous) function $K = K(u, v)$ is defined to be the quotient of the determinants of the second and first fundamental forms of this embedding of (1), then the resulting set function (10) satisfies (9*). The first of these two facts follows by observing that (8*) is an integrated form of Liouville's representation of the Gaussian curvature of a C''-metric (cf. [1], p. 123), while the second fact follows from the circumstance that the embedded form of (9*) is identical with formula (7), p. 759, of [6], a formula which holds on surfaces $X = X(u, v)$ of class C''; cf. [18], pp. 42-44 and [13], p. 135 or [6], p. 760.

2. In order to deal with questions involving the normal image on a sphere, or with the third fundamental form, of a surface in a manner which avoids the usual unnatural restrictions of differentiability (restrictions which lack a direct geometrical significance), the case $K_0 = 1$ of the following theorem will be needed:

(i) *If a C'-metric* (1) *on D possesses a constant curvature* (*in the sense of* (10), *that is, if the set function* (7), *where $E = E(J)$, is representable in the form*

$$(11) \qquad \tau(E) = K_0 \iint_E (\det g_{ik})^{\frac{1}{2}} du dv,$$

where $K_0 = \text{const.} \gtreqless 0$), *then there exists, near every point* (u, v) *of D, a transformation* (5) *which is of class C'' and of non-vanishing Jacobian, and which transforms* (1) *into the standard analytic form*

$$(12) \qquad (du^{*2} + dv^{*2})/f^2, \quad \text{where } f = 1 + (u^{*2} + v^{*2})K_0/4.$$

This theorem (i), which is a refinement (for the two-dimensional case) of the characterization of metrics of constant curvature due to Weingarten (cf. [2]), is contained between the lines of [19]. In fact, the situation is as follows: In order to prove the theorem in its preceding formulation, it is sufficient to show that, if the total curvature (7) of a C'-metric satisfies (11), then Weingarten's system of three linear differential equations

$$(13) \qquad \phi_{ik} - \Gamma^n{}_{ik}(u^1, u^2)\phi_n + K_0 g_{ik}(u^1, u^2)\phi = 0$$

(in which the functions (4) are just continuous and the subscripts of ϕ denote partial differentiations with respect to $u^1 = u$ and $u^2 = v$) is "total." By this is meant that (13) possesses a (unique) solution $\phi = \phi(u, v)$ for which the function ϕ and its first partial derivatives ϕ_1, ϕ_2 reduce to arbitrarily given values, ϕ^0 and $\phi_1{}^0$, $\phi_2{}^0$, at an arbitrary point (u^0, v^0) of D. For, if this is assured, then the above theorem (i) follows by the arguments used in [19], Sections 4 and 7. But the system (13) of three equations of second order will be "total" if it is "total" when written in the form of six equations of first order,

$$(14) \qquad \partial\phi/\partial u^i = \psi_i, \qquad \partial\psi_i/\partial u^k = \Gamma^n{}_{ik}\psi_n - K_0 g_{ik}\phi.$$

In view of theorem (II) in [6], this requires that the system (14) should satisfy the set of the integrability conditions to which the last two formula lines of theorem (II) in [6], p. 761, reduce in the present case, represented by (14). Since (14) is a homogeneous system in three unknown functions ϕ, ψ_1, ψ_2 and two independent variables, there are 9 ($= 3^2$) such integrability conditions. A direct calculation shows that one of these reduces to the identity $0 = 0$; two are satisfied by virtue of $g_{ik} = g_{ki}$ and $\Gamma^j{}_{ik} = \Gamma^j{}_{ki}$; two are equivalent to the conditions

$$\int_J K_0 g_{ik} du^k = \iint_E K_0(\partial g_{i2}/\partial u^1 - \partial g_{i1}/\partial u^2)\, du^1 du^2, \quad \text{where } i = 1, 2,$$

which are satisfied if and only if K_0 is a constant; and the last four are represented by the equations

$$\int_J \Gamma^j_{in} du^n = \iint_E (\Gamma^n_{i1}\Gamma^j_{n2} - \Gamma^n_{i2}\Gamma^j_{n1} + (-1)^i g^{ij} K_0 g^2)\, du^1 du^2,$$

where $i, j = 1, 2$ and $(g^{ik}) = (g_{ik})^{-1}$. These four relations are equivalent, by virtue of the Lemma in [6], p. 761, to four others in which the line integral on the left is replaced by $\int_J g^{-1} g_{jk} \Gamma^j_{in} du^n$, where $i, k = 1, 2$. Among these last four integral conditions, two are trivially satisfied and two reduce to (9), provided that the C'-metric (1) has the curvature K_0. Since this is precisely the assumption of (i), the proof of (i) is complete.

The proof of (i) has the following consequence:

(i bis) *Let* (1) *be a C'-metric on a simply connected domain D; Γ^j_{ik} the Christoffel symbols of the second kind belonging to* (1); *finally, K_0 a continuous function on D. For every point (u_0, v_0) of D and every set of three numbers ϕ^0, $\phi_1{}^0$, $\phi_2{}^0$, there exists a solution $\phi = \phi(u, v)$ of class C'' on D of* (13) *satisfying $\phi(u_0, v_0) = \phi^0$, $\phi_1(u_0, v_0) = \phi_1{}^0$, $\phi_2(u_0, v_0) = \phi_2{}^0$ if and only if K_0 is a constant and the metric* (1) *has a curvature K, which is the constant $K = K_0$.*

The proof of (II) in [6], pp. 763-765, on which the proof of (i), (i bis) is based, shows that if Γ^j_{ik}, g_{ik} are arbitrary continuous functions in (13), then (whether or not (13) is "total") (13) has at most one solution $\phi(u, v)$ of class C'' satisfying given initial conditions $\phi(u_0, v_0) = \phi^0$, $\phi_1(u_0, v_0) = \phi_1{}^0$, $\phi_2(u_0, v_0) = \phi_2{}^0$.

Part II.

3. A set S of points in the euclidean space $X = (x, y, z)$ will be called a surface of class C^n, where $n \geqq 1$, if there exist some (u, v)-domain D and some vector function $X(u, v)$ of class C^n on D such that the vector product $[X_1, X_2]$, where $X_1 = \partial X/\partial u$, $X_2 = \partial X/\partial v$, does not vanish and $X = X(u, v)$ is a one-to-one mapping of D onto S. The vector function $X = X(u, v)$ will be said to be a C^n-parametrization of S. The classes C^1, C^2, C^3 will be denoted by C', C'', C'''.

Suppose that S is of class C'', and let $X = X(u, v)$ be a C''-para-

metrization of S. Then the unit normal vector

(15) $$N = [X_1, X_2]/|[X_1, X_2]| \qquad ([X_1, X_2] \neq 0)$$

is a function $N(u, v)$ of class C' on D. If the binary symmetric matrices

(16) $\alpha = (a_{ik})$, (17) $\beta = (b_{ik})$, (18) $\gamma = (c_{ik})$

are defined by

(19) $a_{ik} = X_i \cdot X_k$, (20) $b_{ik} = -N_i \cdot X_k$, (21) $c_{ik} = N_i \cdot N_k$,

then $\alpha = \alpha(u, v)$ is of class C', while $\beta = \beta(u, v)$ and $\gamma = \gamma(u, v)$ are continuous (on D) and

(22) $|dX|^2 = a_{ik}du^i du^k$, (23) $-dX \cdot dN = b_{ik}du^i du^k$,

(24) $|dN|^2 = c_{ik}du^i du^k$

are, respectively, the first, second, third fundamental forms on

$$S\colon X = X(u^1, u^2),$$

where $u^1 = u$, $u^2 = v$.

For reasons which will become obvious in a moment, the coefficients of the second fundamental form are defined by (20), and not by the formula $b_{ik} = N \cdot X_{ik}$, involving the second derivatives of X. However, the possibility of so defining them (for the case at hand) shows that β is a symmetric matrix.

The Gaussian and mean curvatures, K and H, are defined by

(25) $K = \det(\beta\alpha^{-1})$, (26) $H = \frac{1}{2}\mathrm{tr}(\beta\alpha^{-1})$

and are continuous functions on D. The reciprocal α^{-1} of (16) exists, since the parenthetical assumption of (15) means that

(26) (22) is positive definite.

On the other hand,

(27) (24) is positive definite at non-parabolic points only,

that is, if and only if $K \neq 0$. In fact, (27) follows from (24), (22) and from the identity

(28) $$[N_1, N_2] = K[X_1, X_2], \qquad ([X_1, X_2] \neq 0),$$

while (28) follows from (25) and (19)-(20) if use is made of Weingarten's derivation formulae

(29) $$N_j = -b_{ji}a^{ik}X_k, \quad \text{where } j = 1, 2 \text{ and } (a^{ik}) = \alpha^{-1}.$$

Both (28) and (29) hold also at parabolic points (u, v), points at which (24) is positive semi-definite. Even at the latter points, the matrix of (24) is uniquely determined by the matrices of (22) and (23), since

$$(30) \qquad \gamma = \beta a^{-1} \beta.$$

In fact, (30) follows if (29) is substituted into (21) and then use is made of (19) and (20); cf. [21], p. 372.

The classical representation of the third fundamental form in the terms of the first and the second is not (30) but

$$(31) \qquad \gamma = -Ka + 2H\beta$$

with (25)-(26). The standard proof of (31) is open to objections, partly because it excludes umbilical points (points (u, v) at which $H^2 = K$) and which can form a complicated (u, v)-set even if S is of class C^∞, but mainly because it is valid only if S has a C''-parametrization $X = X(u, v)$ in which $u =$ const. and $v =$ Const. are lines of curvature, whereas all that is assumed now is that S has *some* C''-parametrization. In order to verify (31) under the latter assumption alone, note that, if ϵ denotes the unit matrix, then, since the definitions (25)-(26) reduce the characteristic equation $\det(s\epsilon - a^{-1}\beta) = 0$ of $a^{-1}\beta$ to $s^2 - 2Hs + K = 0$, the latter equation must be satisfied by $s = a^{-1}\beta$ (Hamilton-Cayley). If the resulting matrix relation is multiplied by a from the right, it follows that (31) is equivalent to (30).

Incidentally, (30) and (25) imply (27) and also show that, at every point (u, v) of D, any of the three assumptions $K = 0$, $\det \beta = 0$, $\det \gamma = 0$ is equivalent to the other two. This implies that the first fundamental form is uniquely determined by the second and the third, in any (u, v)-domain not containing parabolic points, since (30) can be written in the form

$$(32) \qquad a = \beta\gamma^{-1}\beta \text{ if (and only if) } K \neq 0.$$

It also follows that, under the assumption of (32), the relations (25), (26) can be written as

$$(33) \quad 0 \neq K = \det(\gamma\beta^{-1}), \qquad (34) \quad H = \tfrac{1}{2}\operatorname{tr}(\gamma\beta^{-1})$$

(Weingarten).

4. Let a sufficiently small (u, v)-domain D be mapped on a (u^*, v^*)-domain D^* by a transformation (5) of class C' and of non-vanishing Jacobian. Then, if $X = X(u, v)$ is a C''-parametrization (on D) of a surface S of class C'', the parametrization

(34) $$X = X(u^*; v^*) \equiv X(u(u^*, v^*), v(u^*, v^*))$$

of S (on D^*) will in general be of class C' only. Nevertheless, *the unit normal vector*

(35) $$N = N(u^*; v^*),$$

defined by that analogue of (15) *in which* X_i *is replaced by* $\partial X/\partial u^{*i}$, *is a function of class* C' *on* D^* (*instead of being just continuous*). For, on the one hand, $N(u^*; v^*) = \pm N(u(u^*, v^*), v(u^*, v^*))$, where $\pm = \operatorname{sgn} \partial(u, v)/\partial(u^*, v^*)$ and, on the other hand, both $N(u, v)$ and $u = u(u^*, v^*)$, $v = v(u^* v^*)$ are of class C'.

Thus those analogues of (20), (21), (22) in which X_i, N_k are replaced by $\partial X/\partial u^{*i}$, $\partial N/\partial u^{*k}$ lead to continuous matrix functions

(36) $\alpha^* = (a^*_{ik})$, (37) $\beta^* = (b^*_{ik})$, (38) $\gamma^* = (c^*_{ik})$

of (u^*, v^*) on D^*. It is clear that α^* and γ^* are symmetric matrices. In view of $X(u, v) = X(u^*; v^*)$ and $N(u, v) = N(u^*; v^*) \operatorname{sgn} \partial(u, v)/\partial(u^*, v^*)$, the transformation rules $\alpha \rightarrow \alpha^*$, $\beta \rightarrow \beta^*$, $\gamma \rightarrow \gamma^*$ are identical with the standard transformation rules when (5) is a substitution of class C''; more precisely,

(39_α) $\alpha^* = \phi' \alpha \phi$, ($39_\gamma$) $\gamma = \phi' \gamma \phi$,

though what would correspond to (39_β) must be replaced by

(40) $$\beta^* = \pm \phi' \beta \phi, \text{ with } \pm = \operatorname{sgn} \det \phi,$$

where ϕ is the Jacobian matrix

(41) $$\phi = (\partial u^i/\partial u^{*k})$$

and ϕ' is the transpose of ϕ. The symmetry of β and the transformation rule (40) show that β^* is symmetric. In view of the definitions of α^*, β^*, γ^*, the analogues of the relations (25)-(34), obtained by replacing α, β, γ, X_i, N_k by α^*. β^*, γ^*, $\partial X/u^{*i}$, $\partial N/\partial u^{*k}$, remain valid.

5. Let a parametrization $X = X(u^*, v^*)$ of a surface S of class C'' be called a *normal C'-parametrization* if $X = X(u^*, v^*)$ is of class C' (in contrast to certain other parametrizations $X = X(u, v)$ of S, in which $X(u, v)$ is of class C'') and if the normal $N = N(u, v)$ gives a C''-parametrization of a portion of the unit sphere $|N| = 1$. A normal C'-parametrization $X = X(\lambda, \mu)$ of S will be called a *spherical C'-parametrization* if the parameters λ, μ satisfy $\lambda^2 + \mu^2 < 1$ and are two of the three direction cosines of N; for example,

(42) $$N = (\lambda, \mu, \nu)$$
where

(43) $\nu = (1 - \rho^2)^{\frac{1}{2}}$, (44) $\rho = (\lambda^2 + \mu^2)^{\frac{1}{2}}$

(or $N = (\nu, \lambda, \mu)$ or $N = (\lambda, \nu, \mu)$).

(ii) *A surface S of class C'' has a normal (and/or spherical) C'-parametrization $X = X(u^*, v^*)$ if and only if S is free of parabolic point, that is,*

(45) $K \neq 0$ (i. e., $\det \beta \neq 0$ and/or $\det \gamma \neq 0$).

(ii bis) *A spherical parametrization $X = X(u^*, v^*)$ of S is of class C'' if and only if S is of class C'''* (that is, if and only if S has, besides a C''-parametrization $X = X(u, v)$ which is assumed, some C''-parametrization $X = X(u', v')$ also).

It is understood that S is always meant to be "sufficiently small," that is, that the assertions are local, in the sense of referring to some vicinity of a given point of S.

In order to prove (ii), let $X = X(u, v)$ be a C''-parametrization of S. Then it is clear from (28) that $N = N(u, v)$ is a C'-parametrization of (part of) the sphere $|N| = 1$ if and only if $K \neq 0$ on S. If $K \neq 0$, hence $[N_1, N_2] \neq 0$ in view of (15), then there exists a transformation $(u, v) \to (u^*, v^*)$, which is of class C' and of non-vanishing Jacobian, and in which (u^*, v^*) is (λ, μ) or (μ, ν) or (λ, ν), finally, which transforms $N(u, v)$ into (42)-(43). Clearly, $X = X(u^*, v^*)$ is a normal C'-parametrization; in fact, it is a spherical C'-parametrization of S. Hence, $K \neq 0$ is sufficient for the existence of spherical C'-parametrizations. Incidentally, this argument makes it clear that spherical parametrizations of a given class $C', C'', \cdots$ exist if and only if normal parametrizations of the same class do.

The necessity of $K \neq 0$ is clear from the definition of normal parametrizations, in which it is required that $N(u^*, v^*)$ be a C''-parametrization of a portion of the unit sphere; in particular, that the vector product of $\partial N/\partial u^*$ and $\partial N/\partial v^*$ be distinct from 0. Since the analogue of (28) holds in C'-parametrizations $X = X(u^*, v^*)$ of a surface of class C', it follows that $K \neq 0$ when S has a normal C'-parametrization. This proves (ii).

The proof of the "if" part of (ii bis) is clear from the proof of the first part of (ii). For if $X = X(u, v)$ is a C'''-parametrization of S, then $N = N(u, v)$ is a C''-parametrization of part of the sphere $|N| = 1$ and the above-described transformation $(u, v) \to (u^*, v^*)$ is of class C'' and leads to a spherical C''-parametrization. Since a spherical parametrization is

derived from another (when two exist) by an analytic transformation of the parameters, all (one or three) spherical parametrizations are of class C'' when S is of class C'''. The converse, that is, the "only if" part of (ii bis), is contained in the following lemma as a particular case, $u^* = u$, $v^* = v$ and $n = 2$.

LEMMA. *If a surface S, of class C^n for a fixed $n \geqq 1$, possesses a C^n-parametrization $X = X(u, v)$ in which the normal $N = N(u, v)$ becomes a function of class C^n, then S is a surface of class C^{n+1}* (that is, S has some C^{n+1}-parametrization $X = X(u', v')$, say $z = z(x, y)$, where $(x, y, z) = X$).

Since this lemma is known (cf. [8], p. 163, the last paragraph of Section 14), the proof of (ii bis) is now complete.

Remark. A corollary of (ii) is the following statement: If S is a surface of class C'', then it possesses C'-parametrizations $X = X(u^*; v^*)$ in terms of which the normal $N = N(u^*; v^*)$ is a function of class C'', too, provided that (45) holds on S. Since no mention is made now of normal parametrizations, that is, since it is not required that $[\partial N/\partial u^*, \partial N/\partial u^*] \neq 0$, it is natural to ask whether the proviso (45) can now be omitted. The answer proves to be in the negative. In fact, an example to this effect is supplied by every surface S which is of class C'' without being of class C''' (cf. the case $n = 2$ of the above Lemma) and which is a torse containing no flat points (that is, if $K \equiv 0 \neq H$ on S). Cf. Part V of [8].

Part III.

6. The purpose of the following theorems is to remove from the Codazzi equations belonging to the third fundamental form (Weingarten) unnecessary assumptions of differentiability which are implicit in the standard treatment of this problem (cf. [1], pp. 232-234).

(iii) *If S is a surface of class C'' having no parabolic points and if $X = X(u, v)$ is a normal C'-parametrization of S on a simply connected domain D, then, in these parameters, the second and third fundamental forms*

(46) $\quad b_{ik}(u^1, u^2)\, du^i du^k,$ $\qquad$ (47) $\quad c_{ik}(u^1, u^2)\, du^i du^k,$

given by (23), (24) *and* (15), *have the following properties:*

(†) *The form* (47) *is a C'-metric possessing the constant curvature $K_0 = 1$; the coefficients b_{ik} in* (46) *are of class C^0* (*continuous*) *and satisfy*

$$\det b_{ik} \neq 0; \tag{48}$$

finally, if the Γ^j_{ik} *are the Christoffel symbols of the second kind belonging to* (47), *then both relations*

$$\int_J b_{ik} du^k = \iint_E (\Gamma^j_{i1} b_{j2} - \Gamma^j_{i2} b_{j1}) du^1 du^2 \qquad (i = 1, 2) \tag{49}$$

are identities in J, *where* $E = E(J)$ *denotes the interior of any positively oriented, piecewise smooth Jordan curve* J *contained in* D.

In the classical statement of this theorem, it is (tacitly) supposed that S is of class C''''. If $X = X(u, v)$ is a C''''-parametrization of S, the matrices α, β, γ, defined by (16)-(21), are of class C''', C'', C'', respectively. In such a parametrization, the condition that (47) have the constant curvature $K_0 = 1$ is then expressible in terms of the second derivatives of the c_{ik}, while the "Codazzi" relations (49) are expressible in terms of the first derivatives of the b_{ik} (and c_{ik}). Thus (iii) improves on the classical theory by two degrees of differentiability—one resulting from the choice of normal parameters, and another from the fact that the condition on the curvature of (47) and the Codazzi relations (49) are used in an "integrated" form.

In a certain sense, conditions (†) of (iii) characterize the second and third fundamental forms in normal parametrizations of a surface of class C''. As does (iii), the following version of the "converse" of (iii) improves on the classical version by two degrees of differentiability.

(iii*) *Let the quadratic differential forms* (46), (47) *have the properties* (†) *of* (iii) *on a simply connected domain* D. *Then there exists a surface* S *of class* C'' *having on* D *a normal* C'*-parametrization* $X = X(u, v)$ *in which* (46) *and* (47) *become the second and third fundamental forms, respectively; that is,* (15) *is of class* C'' *and* (23), (24) *hold on* D. *The surface* S *is uniquely determined* (*up to movements of the Euclidean* X*-space*).

The proofs of (iii), (iii*) will only be sketched. They are modifications of the proofs in the classical case of higher differentiability (cf. [1], pp. 232-235). The necessary modification of those proofs is similar to the modification of the proof of Bonnet's theorem (characterizing the first and second fundamental forms), used in [6], pp. 761-762; the main point being that the standard theorem on total systems in the classical proofs is replaced by the theorem (II) of [6], pp. 760-761.

If $N = N(u, v)$, where $(u, v) = (u^1, u^2)$ is on a simply connected domain D, is a C''-parametrization of a portion of the sphere $|N| = 1$, then the derivation formulae of Gauss for the sphere are

(50) $$N_{ik} = \Gamma^j_{ik} N_j - c_{ik} N,$$

where c_{ik} is defined by (22) and Γ^j_{ik} are the Christoffel symbols belonging to (47). In view of (30), the Weingarten derivation formulae (29) for S can be written as

(51) $$X_k = - b_{ki} c^{ij} N_j, \quad \text{where} \quad k = 1, 2,$$

if $(c^{ij}) = \gamma^{-1}$, and if (45) and (48) hold.

If (50)-(51) is considered as a linear system for the unknowns X, N, N_1, N_2, then theorem (II) in [6,] pp. 760-761, shows that this system is "total" if and only if (47) has the constant curvature $K_0 = 1$ and the relations (49) hold as an identity in J. In fact, X, N, N_1, N_2 can be considered as scalars, since (50)-(51) consists of three identical systems, one for each component of the vectors. In this case (of four unknown functions and two independent variables), there are 16 ($= 4^2$) integrability functions to be treated. When N is a scalar, (50) is identical with the Weingarten equation (13) if g_{ik} in the latter equation is replaced by c_{ik} and K_0 by 1. Hence, 9 of the 16 integrability conditions have been dealt with in the proof of (i), (i bis) and are satisfied if and only if (47) has the constant curvature $K_0 = 1$. Of the 7 remaining conditions, a direct calculation shows that 5 are trivial (of the type $0 = 0$) and 2 reduce to the relations (49).

Let S satisfy the conditions of (iii). Then the given parametrization $X = X(u, v)$ of S supplies an N and an X satisfying (50), (51). Every surface obtained from S by Euclidean movements also gives a solution X, N of (50), (51). By considering (50), (51) as scalar equations (say, as equations involving the first component), it is clear that from these solutions it is possible to construct solutions in which X, N, N_1, N_2 take arbitrary values at a given point of D, since (50)-(51) are linear and homogeneous. Hence, the necessity of the integrability conditions follows from theorem (II) of [6]. Since S has no parabolic points, (48) holds. Thus (46), (47) satisfy (†). This proves (iii).

In order to prove (iii*), note that if (46), (47) satisfy (†), then the "total" character of (50)-(51) leads to solutions X, N of class C', C'', respectively, uniquely determined by initial conditions. As in the classical case, a solution X, N satisfies (15), (19), (20) and (21), provided that the initial conditions of X, N, N_1, N_2 do. Finally, the Lemma (above) shows that there exists a surface S of class C'' for which X is a normal C'-parametrization. This proves (iii*).

A corollary of the proof of (iii*) is the fact that if conditions (†) are unaltered except that, in addition, it is supposed that (b_{ik}) is of class C',

then there result surfaces S of class C''' (instead of class C''), for which (46), (47) become the second and third fundamental forms in a normal C''-parametrization of S. The fact that no additional smoothness hypothesis is made on (47) seems curious at first glance. Actually, (i) shows that there is no loss of generality in assuming that (47) is analytic. A more inclusive consequence of the proof of (iii*) is as follows:

(iii$^*_n$) *If the forms* (46), (47) *are of class C^n (or C^{n-1}), C^n respectively, where $n \geqq 1$, and have the properties* (†) *of* (iii) *on a simply connected domain D, then there exists a surface S of class C^{n+2} (or C^{n+1}) having on D a C^n- (or C^{n-1}-) parametrization $X = X(u, v)$ in which* (15) *is of class C^{n+1} and for which* (20), (21) *hold.*

Part IV.

7. The point coordinates $X = (x, y, z)$ of S will now be replaced by its plane coordinates or its *supporting function*

$$w = X \cdot N \tag{52}$$

(which, according to (15), is defined even if S is of class C' only). In view of (iii), the result of the Appendix of [21], pp. 374-376, can be stated as follows:

(iv) *If S is a surface of class C'' without parabolic points and if $X = X(\lambda, \mu)$ is a spherical C'-parametrization of S, then the supporting function*

$$w = w(\lambda, \mu) = X(\lambda, \mu) \cdot N(\lambda, \mu) \tag{53}$$

is of class C'' (even though $X(\lambda, \mu)$ is of class C', and cannot be of class C'' unless S happens to be of class C''').

The proof of (iv) implies the following extension of (iv):

(iv$_n$) *The assertion of* (iv) *remains true if the respective classes C'', C' are replaced by C^n, C^{n-1}, where $n \geqq 2$.*

If S is of class C'' without parabolic points, then, by (iii), it posseses spherical C'-parametrizations $X = X(\lambda, \mu)$. After a suitable rotation of the X-space, it can be supposed that, in such a parametrization, (42)-(44) hold, where $\rho^2 < 1$. In this case, the notation

$$\lambda = \lambda^1, \qquad \mu = \lambda^2 \tag{54}$$

will be used and the letters a, b, c in (22), (23), (24) will be changed to g, h, f; that is, the three fundamental forms on S in terms of the (spherical) parameters (54) will be denoted by

$$(55) \quad |dX|^2 = g_{ik} d\lambda^i d\lambda^k; \qquad (56) \quad -dX \cdot dN = h_{ik} d\lambda^i d\lambda^k;$$

$$(57) \quad |dN|^2 = f_{ik} d\lambda^i d\lambda^k$$

(the representation $-X_i \cdot N_k = -X_k \cdot N_i$ of h_{ik} in (56) can, as in (20), be replaced by $X_{ik} \cdot N = h_{ik}$ only if not merely the surface S but also its C'-parametrization $X = X(\lambda, \mu)$ is of class C''; cf. the remark following (24) above).

Substitution of (42), (43), (44) into (57) shows that the coefficients f_{ik} of the third fundamental form are

$$(58) \quad f_{11} = (1 - \mu^2)/\nu^2, \qquad f_{12} = \lambda\mu/\nu^2, \qquad f_{22} = (1 - \lambda^2)/\nu^2,$$

where $\nu^2 = 1 - \lambda^2 - \mu^2$. Since the function (53) is of class C'', it has (continuous) second covariant derivatives

$$(59) \quad \nabla_{ik} w \equiv w_{ik} - \Gamma^j{}_{ik} w_j \qquad (i, k = 1, 2),$$

with reference to the Christoffel coefficients of the metric (57) defined by (58). A simple calculation shows that (59) becomes

$$(60) \quad \nabla_{11} w = w_{11} - (1 - \mu^2)\nu^{-2}\omega, \qquad \nabla_{12} w = w_{12} - \lambda\mu\nu^{-2}\omega,$$
$$\nabla_{22} w = w_{22} - (1 - \lambda^2)\nu^{-2}\omega,$$

where

$$(61) \quad \omega = \lambda w_1 + \mu w_2 \quad \text{and} \quad \nu^2 = 1 - \lambda^2 - \mu^2.$$

The subscripts of w in (59), (60), (61) (and later on) denote partial differentiations with respect to $\lambda = \lambda^1$ and $\mu = \lambda^2$.

In addition to (59), the following notations (cf. [1], p. 87) will be used: $\Delta_{22} w$ will denote the Monge-Ampère operator det $\{(f_{ik})^{-1}(\nabla_{ik} w)\}$ and $\Delta_2 w$ will be the Laplacian operator with respect to (57). In view of (58), this means that

$$(62) \quad \Delta_{22} w = \nu^2 \det \nabla_{ik} w, \qquad (63) \quad \Delta_2 w = \nu^2 \Delta^2 w,$$

where

$$(64) \quad \Delta^2 w = (1 - \lambda^2) w_{11} - 2\lambda\mu w_{12} + (1 - \mu^2) w_{22}.$$

Under the assumptions of (iv), the mean curvature H is continuous and the Gaussian curvature K is continuous and non-vanishing. In terms of the notations (62)-(64), the product and the sum of the principal radii of

curvature, that is, $1/K$ and $2H/K$, are given by Weingarten's formulae [17]

(65) $1/K = \Delta_{22}w + w\Delta_2 w + w^2$, (66) $2H/K = \Delta_2 w + 2w$

(cf. *loc. cit.*, p. 259), while the coefficients of the second form (56) follow from (42), (56) and (53),

$$-h_{ik} = w_{ik} + wf_{ik} \tag{66}$$

(*ibid.*, top of p. 259). The coefficients g_{ik} of the first fundamental form can be represented in terms of (58) and (52) by inserting (65) and (66) into

$$Kg_{ik} = 2Hw_{ik} + (1 + 2wH)f_{ik}. \tag{67}$$

In fact, (66) shows that (67) follows from (30) and (31), where α, β, γ are the matrices of (55), (56), (57), respectively.

Finally, the spherical parametrization $X = X(\lambda, \mu)$ of S can be obtained in terms of its supporting function $w = w(\lambda, \mu)$ and of (42), (58) as follows:

$$X = wN + f^{ik}w_i N_k, \quad \text{where} \quad (f^{ik}) = (f_{ik})^{-1} \tag{68}$$

(*ibid.*, p. 256 and p. 277).

Theorem (iv) and its consequence (68) have the following converse:

(iv*) *On a small domain of the circle* $\lambda^2 + \mu^2 < 1$, *let* $w = w(\lambda, \mu)$ *be a function of class* C'' *satisfying*

$$\Delta_{22}w + w\Delta_2 w + w^2 \neq 0 \tag{69}$$

(*cf.* (60)-(64)) *and let* $N = N(\lambda, \mu)$ *denote the unit vector defined by* (42)-(43). *Then there exists a unique surface* S *of class* C'' *having a spherical* C'-*parametrization* $X = X(\lambda, \mu)$ *with respect to which* $\pm N(\lambda, \mu)$ *is the normal* (15) *and* $\pm w(\lambda, \mu)$ *is the supporting function* (53) (*where* $\pm$ *is the signature of the expression* (69)).

It is clear from (65) that the condition (69) cannot be omitted in (iv*). The uniqueness of S is clear from the derivation of (68).

In order to prove (iv*), let $X = X(\lambda, \mu)$ be defined by (68) (and (58)). Clearly, (68) is a function of class C'. In view of (42), (58), (60) and (61), the vector (68) can be written as $X = (w - \omega)N + M$, where M is the "vector" $(w_1, w_2, 0)$. It is seen from (60) that $w_k - \omega_k = -\lambda^j w_{jk}$, where $k = 1, 2$; so that

$$X_k = (w - \omega)N_k - w_{jk}\lambda^j N + M_k.$$

Since the scalar product $M_k \cdot N$ is $w_{jk}\lambda^j$, it follows that X_1 and X_2 are orthogonal to N. Actually, the vector product $[X_1, X_2]$ is $\nu^{-1}N$ times the

expression in (69). (For the purposes at hand, it can be supposed that $(\lambda, \mu) = (0, 0)$ is a point of D, and it is then sufficient to verify the last assertion at that point.) Hence, when (69) holds, $X = X(\lambda, \mu)$ is a C'-parametrization of a surface S having $\pm N$ as its normal (15), where $\pm$ is the signature of the expression (69). It is clear from (68) that $\pm w$ is the supporting function of S. The Lemma (Section **5** above) implies that S is a surface of class C'', since both $X(\lambda, \mu)$ and $\pm M(\lambda, \mu)$ are of class C'.

It is clear from the proof that (iv*) can be extended as follows:

(iv*$_n$) *The assertion* (iv*) *remains valid if the classes* C', C'' *are replaced by* C^n, C^{n+1}, *respectively, where* $n \geqq 1$.

Part V.

8. Theorem (iii*) concerns the "embedding" of a given pair of second and third fundamental forms. In what follows, there will be considered the embedding of a given (positive definite) third fundamental form and of either a given Gaussian curvature K ($\neq 0$) or of a given mean curvature H; in other words, the embedding of a K ($\neq 0$) or H given as a function of the normal N.

(v) *On a small simply connected* (u, v)*-domain* D, *let* (47) *be, for some* $n \geqq 1$, *a* C^n*-metric having the constant curvature* $K_0 = 1$. *Let* $\phi(u, v)$ *be a function of class* C^n *on* D. *Then there exist surfaces* S *of class* C^{n+1}, *having a normal* C^n*-parametrization* $X = X(u, v)$ *on* D, *with respect to which* (47) *is the third fundamental form* (24) *and the given* $\phi(u, v)$ *is any of the following functions:*

(v^1) *the mean curvature* $H(u, v)$;

(v^2) *the Gaussian curvature* $K(u, v)$, *provided that* $\phi \neq 0$;

(v^3) *the ratio* $2H/K$, *the sum of the principal radii of curvature.*

The auxilary condition $\phi \neq 0$ is necessary in (v^2), since it is assumed that (47) is positive definite. Since the metric (12), where $K_0 = 1$, and the form in (57) with coefficients (58) are equivalent by virtue of an analytic transformation $(u^*, v^*) \rightarrow (\lambda, \mu)$, it follows from (i) that it can be supposed that $(u, v) = (\lambda, \mu)$ and that the given coefficients c_{ik} are those given by (58). In fact, the C''-transformation $(u, v) \rightarrow (\lambda, \mu)$ defined by $(u, v) \rightarrow (u^*, v^*) \rightarrow (\lambda, \mu))$ will leave the assumptions of (†) on (47) and $\phi(u, v)$ unchanged. It can also be supposed that a given point (u, v) corresponds to $(\lambda, \mu) = (0, 0)$.

If $w = w(\lambda, \mu)$ is the supporting function of a surface S (the existence of which is to be proved), then, corresponding to (v^1), (v^2) or (v^3), the function w satisfies the respective partial differential equation

$$\Delta_2 w + 2w - 2(\Delta_{22} w + w\Delta_2 w + w^2)\phi(\lambda, \mu) = 0, \tag{70}$$

$$\Delta_{22} w + w\Delta_2 w + w^2 - 1/\phi(\lambda, \mu) = 0, \tag{71}$$

$$\Delta_2 w + 2w - \phi(\lambda, \mu) = 0; \tag{72}$$

cf. (65), (66). On the other hand, (iv^*), (iv^*_n) show that the embedding theorems (v^1), (v^2), (v^3) follow if it is verified that each of the partial differential equations (70), (71), (72) has, on a vicinity of $(\lambda, \mu) = (0, 0)$, solutions $w = w(\lambda, \mu)$ of class C^{n+1} satisfying (69).

Ad (72). It is clear from (63), (64) that the equation (72) is linear in the first and the second derivatives of w and is of elliptic type. Since ϕ is of class C^n, where $n \geqq 1$, it follows that (72) has solutions $w = w(\lambda, \mu)$ of class C^{n+1} (in a sufficiently small vicinity of $(\lambda, \mu) = (0, 0)$); cf. [15], pp. 91-92, and Section **9** below. Furthermore, solutions of (72) can be so chosen that (69) holds.

Remark. Since (72) is a linear, elliptic, partial differential equation, the above proof is valid if, instead of assuming that ϕ is of class C^n, it is only assumed that ϕ has $(n-1)$-st order partial derivatives which satisfy a uniform Hölder condition. When $n = 1$, this means that ϕ satisfies a uniform Hölder condition. On the other hand, it is indicated by considerations analogous to those concerning the Poisson and related equations (cf. [20]), that there exist continuous functions $\phi(\lambda, \mu)$ for which (72) has no solution on any (λ, μ)-domain. Hence, if $\phi(P)$ is an arbitrary continuous function of the position P on the sphere, it is unlikely that there exists a (closed) surface X of class C'' which has a normal image covering the sphere in a one-to-one manner and which satisfies $2H/K = \phi$, where the normal of X corresponds to the point P of the unit sphere. On the other hand the uniqueness of such a surface, if it exists, is known (Christoffel; cf. [12], p. 551).

Ad (70) *and* (71). The equations under consideration are of the Monge-Ampère type

$$A + Br + Cs + Dt + E(rt - s^2) = 0,$$

which is elliptic or hyperbolic according as

$$C^2 - 4BD + 4AE$$

is negative or positive.

In the case of (71), it is seen from (60)-(64) that, at $(\lambda, \mu) = (0, 0)$, the coefficients are $A = w^2 - 1/\phi$, $B = w$, $C = 0$, $D = w$ and $E = 1$. Hence, the expression in the last formula line becomes $-4/\phi$ at $(\lambda, \mu) = (0, 0)$. Thus (71) is elliptic or hyperbolic according as the assigned (non-vanishing) curvature ϕ is positive or negative. In the hyperbolic case, a known existence theorem (cf. [11], p. 849) shows that (71) has solutions of class C^{n+1} when ϕ is of class C^n, where $n \geqq 1$. In the elliptic case, the existence of such solutions can be deduced from an analogue of Lemma 2 of [7], p. 557; cf. Section **9** below. (In the case at hand, the condition (69) holds in view of (71)).

In the case of (70), the coefficients at $(\lambda, \mu) = (0, 0)$ are $A = 2\phi w^2 - 2w$, $B = 2\phi w - 1$, $C = 0$, $D = 2\phi w - 1$ and $E = 2\phi$. Hence, the expression in the last formula line becomes -4; so that (70) is of elliptic type near $(\lambda, \mu) = (0, 0)$. If ϕ is of class C^n, then (70) has solutions of class C^{n+1} satisfying (69). This is a consequence of the existence theorem in Section **9** below.

8 bis. Theorems (v^1), (v^2), (v^3) are special cases of an embedding theorem in which there are assigned as the third fundamental form a quadratic differential form (47) having the constant curvature $K_0 = 1$, and a given relation $F(2H/K, 1/K; u, v) = 0$ to be satisfied by $2H/K$ and $1/K$, the sum and the product of the principal radii of curvature. For the sake of simplicity, this general embedding theorem will be considered only when (u, v) is (λ, μ) and the given form (47) has the coefficients $c_{ik} = f_{ik}$ defined by (58). Let $F(U, V; \lambda, \mu)$ be a continuous function in a vicinity of a point $(U^0, V^0; 0, 0)$ and suppose that

(73) $$F(U^0, V^0; 0, 0) = 0, \quad V^0 \neq 0 \quad \text{and} \quad (U^0)^2 \geqq 4V^0.$$

The condition $V^0 \neq 0$ corresponds to the condition $1/K \neq 0$; the last part of (73) corresponds to the inequality $H^2 \geqq K$ and assures that (77) below can be satisfied by some numbers w^0, $w_1{}^0$, $w_{11}{}^0$, $w_{12}{}^0$, $w_{22}{}^0$. In addition, let F possess continuous partial derivatives F_U, F_V satisfying

(74) $$VF_V{}^2 + UF_UF_V + F_U{}^2 \neq 0.$$

In view of (65), (66) the embedding problem

(75) $$F(2H/K, 1/K; \lambda, \mu) = 0$$

depends on the partial differential equation

(76) $$F(\Delta_2 w + 2w, \Delta_{22} w + w\Delta_2 w + w^2; \lambda, \mu) = 0.$$

It is readily verified that, at the point $(U, V; \lambda, \mu) = (U^0, V^0; 0, 0)$, the equation (76) is of hyperbolic or elliptic type (that is, $4\partial F/\partial w_{11}\, \partial F/\partial w_{22} - (\partial F/\partial w_{12})^2$ is negative or positive) according as the expression (74) is negative or positive. By saying that (76) is of the hyperbolic or elliptic type at $(U^0, V^0; 0, 0)$, it is meant that if the left-hand side of (76) is considered to be a function of $(\lambda, \mu, w, w_1, w_2, w_{11}, w_{12}, w_{22})$, then (76) is of the specified type at a point $(0, 0, w^0, w_1{}^0, w_2{}^0, w_{11}{}^0, w_{12}{}^0, w_{22}{}^0)$ satisfying

$$(77)\quad \begin{aligned} U^0 &= 2w^0 + w_{11}{}^0 + w_{22}{}^0, \\ V^0 &= w_{11}{}^0 w_{22}{}^0 - (w_{12}{}^0)^2 + (w^0)^2 + w^0(w_{11}{}^0 + w_{22}{}^0); \end{aligned}$$

cf. the remark following (73).

In order to assure the existence of a surface S, say of class C'', having a spherical C'-parametrization $X = X(\lambda, \mu)$, with respect to which $\pm (\lambda, \mu, \nu)$ is its normal vector and (75) is an identity in (λ, μ), where $H = H(\lambda, \mu)$, $K = K(\lambda, \mu) \neq 0$, additional requirements must be imposed on F. In the elliptic case, it is sufficient to require that $F = F(U, V; \lambda, \mu)$ be analytic with respect to (U, V) for fixed (λ, μ), that F and its partial derivatives of first and second order with respect to U, V satisfy a uniform Hölder condition with respect to their four arguments, and that the third derivatives of F with respect to U, V be continuous as functions of their four arguments); cf. Part VI below. In the hyperbolic cases, it is sufficient to require that $F(U, V; \lambda, \mu)$ be of class C'' with respect to its four variables together; cf. [11], pp. 847-848. In the particular hyperbolic case in which (76) is linear (in w_{11} w_{12}, w_{22}) or, more generally, of Monge-Ampère type, the assumption of the class C'' can be relaxed to the assumption that F is of class C'; cf. [11], pp. 848-849 and pp. 855-864.

Part VI.

9. The theorem referred to above, that on which the proof of (v) depends, is an existence theorem for solutions of an ellpitic partial differential equation on small domains. It is a generalization of the elliptic case of Lemma 2 in [7], p. 557, which is suggested by Picard's theorem [16] on the existence of solutions of linear boundary value problems on small domains and by Lichtenstein's theorem [15], p. 90, on non-linear elliptic partial differential equations involving a small parameter. The theorem in question, to be referred to as (§), is as follows:

(§) *Let $\Phi = \Phi(x, y, z, p, q, r, s, t)$ be a function on some eight-dimen-*

sional domain D which satisfies a uniform Hölder condition of order λ with respect to its eight variables, on every compact subset of D and, when (x, y) is fixed, let Φ be analytic with respect to the six variables (z, p, q, r, s, t). Let the first and second order partial derivatives of Φ with respect to z, p, q, r, s, t satisfy a uniform Hölder condition, of order λ, with respect to the eight variables (x, y, t, p, q, r, s, t), and let the third order partial derivatives of Φ with respect to z, p, q, r, s, t be continuous functions of (x, y, z, p, q, r, s, t). Finally, let

$$4\Phi_r\Phi_t - \Phi_s^2 > 0 \text{ on } S, \tag{78}$$

and let $P_0 = (x_0, y_0, z_0, p_0, q_0, r_0, s_0, t_0)$ be a point of D at which

$$\Phi(x, y, z, p, q, r, s, t) = 0. \tag{79}$$

Then, corresponding to a given neighborhood of P_0, there is a neighborhood of $(x, y) = (x_0, y_0)$ on which there exists a function $z = z(x, y)$ with the properties that z has second order partial derivatives satisfying a uniform Hölder condition of any order $\mu < \lambda$, the function $z(x, y)$ satisfies the partial differential equation (79), *and (x, y, z, p, q, r, s, t) is within the given neighborhood of P_0.*

The proof of (§), based on the method of successive approximations which depends on Korn's inequalities [14] in potential theory, will be essentially the same as that used in [18], pp. 64-68, and [15], pp. 90-98.

Proof of (§). Condition (78) implies that $\Phi_r \neq 0$ on P. Since (79) holds at P_0, it follows that (79) can be written in the form

$$r - \Psi(x, y, z, p, q, s, t) = 0 \tag{80}$$

in a neighborhood of P_0, where Ψ satisfies, on some seven-dimensional neighborhood of $(x_0, y_0, z_0, p_0, q_0, s_0, t_0)$, the smoothness conditions analogous to those satisfied by Φ on D. Consider the analytic partial differential equation

$$r - \Psi(x_0, y_0, z, p, q, s, t) = 0, \tag{81}$$

which is equivalent to $\Phi(x_0, y_0, z, p, r, s, t) = 0$. The equation (81) is satisfied at the point $(z, p, q, r, s, t) = (z_0, p_0, q_0, r_0, s_0, t_0)$. Hence, by the Cauchy-Kowalewski existence theorem, the assignment of analytic Cauchy data $z(x_0, y)$, $p(x_0, y)$ which, for $y = y_0$, reduce to $(z_0, p_0, q_0, s_0, t_0)$, where $q_0 = z_y(x_0, y_0)$, $s_0 = p_y(x_0, y_0)$, $t_0 = z_{yy}(x_0, y_0)$, determines a unique analytic solution $z = \zeta(x, y)$ of (81) in a neighborhood of (x_0, y_0). In particular, $\Phi(x_0, y_0, \zeta, \zeta_x, \cdots, \zeta_{yy}) \equiv 0$.

Put

$$z = \zeta(x, y) + u. \tag{82}$$

Then (79) becomes a partial differential equation for u and can be written in the form

(83) $$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + Fu = \Pi.$$

The coefficients A, B, C, D, E are the respective derivatives $\Phi_r, \Phi_s, \Phi_t, \Phi_p, \Phi_q, \Phi_z$ evaluated at $(x_0, y_0, \zeta, \zeta_x, \zeta_y, \zeta_{xx}, \zeta_{xy}, \zeta_{yy})$; so that A, B, C, D, E, F are analytic functions of (x, y). The function $\Pi = \Pi(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy})$ occurring in (83) is the difference of the left side of (83) and of $\Phi(x, y, \zeta + u, \cdot\cdot\cdot, \zeta_{yy} + u_{yy})$. The meaning of Π is clear from the fact that

$$\Phi(x, y, \zeta + u, \cdot\cdot\cdot, \zeta_{yy} + u_{yy}) - \Phi(x_0, y_0, \zeta, \cdot\cdot\cdot, \zeta_{yy})$$
$$= Au_{xx} + Bu_{xy} + \cdot\cdot\cdot + Fu - \Pi.$$

Thus

(84) $$\Pi(x_0, y_0, 0, 0, \cdot\cdot\cdot, 0) = 0,$$

and Π satisfies a uniform Hölder condition of order λ on a neighborhood of $(x, y, u_x, \cdot\cdot\cdot, u_{yy}) = (x_0, y_0, 0, \cdot\cdot\cdot, 0)$. Also, Π is analytic, for fixed (x, y), in the independent variables $(u, u_x, \cdot\cdot\cdot, u_{yy})$, and has with respect to these variables first and second order partial derivatives which satisfy a uniform Hölder condition on a neighborhood of $(x_0, y_0, 0, \cdot\cdot\cdot, 0)$. In addition, Π has continuous third order derivatives with respect to the variables $(u, u_x, \cdot\cdot\cdot, u_{yy})$. The first order partial derivatives of Π satisfy

(85) $\partial\Pi/\partial w = 0$ for $w = u, \cdot\cdot\cdot, u_{yy}$ at $(x, y, u, \cdot\cdot\cdot, u_{yy}) = (x_0, y_0, 0, \cdot\cdot\cdot, 0)$.

Let $\delta > 0$ be so small that $\zeta(x, y)$ is defined (and analytic) on the closure of

(86) $$\mathfrak{G}_\delta\colon \quad (x - x_0)^2 + (y - y_0)^2 < \delta^2.$$

Let $m > 0$ be chosen so that $\Pi(x, y, u, \cdot\cdot\cdot, u_{yy})$ has the properties enumerated, if (x, y) is in (86) and $(u, \cdot\cdot\cdot, u_{yy})$ is subject to the inequalities

(87) $$|u| \leqq m, \; |u_x| \leqq m, \cdot\cdot\cdot, \; |u_{yy}| \leqq m.$$

If $u(x, y)$ is any function on (86) satisfying a uniform Hölder condition of order λ, let $|u|_\lambda$ denote the least upper bound of the numbers M satisfying both inequalities

(88) $$|u(x, y)| \leqq M, \qquad |u(x + h, y + k) - u(x, y)| \leqq M(h^2 + k^2)^{\frac{1}{2}\lambda}$$

for all (x, y), $(x + h, y + k)$ in (86). If $u(x, y)$ is of class $C''(\lambda)$ on (86) (that is, has second order partial derivatives on (86) which satisfy a uniform Hölder condition of order λ), put

$$\| u \|_\lambda = \max(| u |_\lambda, | u_x |_\lambda, \cdots, | u_{yy} |_\lambda). \tag{89}$$

Let N^0 denote an upper bound of the absolute value of Π and its first order partial derivatives with respect to $u, u_x, \cdots, u_{yy}$ on the product set of (86) and (87). Let $N \geqq N^0$ be an upper bound for the absolute value of the second and third order partial derivatives of Π on the domain specified. Finally, let N_λ denote an upper bound for the numbers M satisfying the inequality

$$| \Pi(x + h, y + k, u, \cdots, u_{yy}) - \Pi(x, y, u, \cdots, u_{yy}) | \leqq M(h^2 + k^2)^{\frac{1}{2}\lambda}, \tag{90}$$

and the corresponding inequalities which result if Π is replaced by any of its first and second partial derivatives with respect to $u, u_x, \cdots, u_{yy}$, for all (x, y), $(x + h, y + k)$ on (86), and $u, \cdots, u_{yy}$ satisfying (87).

For any $\epsilon > 0$, it follows from (84) and (85) that if $\delta > 0$ and $m > 0$ are sufficiently small, then N^0 can be chosen so as to satisfy

$$0 \leqq N^0 < \epsilon. \tag{91}$$

Also, if $0 < \mu < \lambda$ and if $\delta > 0$, $m > 0$ are sufficiently small, then N_μ can be chosen so as to satisfy

$$0 \leqq N_\mu < \delta \tag{92}$$

(in fact, $N_\lambda(2\delta)^{\lambda-\mu}$ is a possible choice for N_μ).

If $u = u(x, y)$ is a function of class $C''(\mu)$ on (86) satisfying (87), then the absolute value of $\Pi(x, y) \equiv \Pi(x, y, u(x, y) \cdots, u_{yy}(x, y))$ does not exceed N^0. Also, $| \Pi(x + h, y + k) - \Pi(x, y) |$ is not greater than the sum of

$$| \Pi(x + h, y + k, u(x + h, y + k), \cdots, u_{yy}(x + h, y + k))$$
$$- \Pi(x, y, u(x + h, y + k), \cdots, u_{yy}(x + h, y + k)) |$$

and

$$| \Pi(x, y, u(x + h, y + k), \cdots, u_{yy}(x + h, y + k))$$
$$- \Pi(x, y, u(x, y), \cdots, u_{yy}(x, y)) |.$$

This sum does not exceed

$$N_\mu(h^2 + k^2)^{\frac{1}{2}\mu} + N^0 | u(x + h, y + k) - u(x, y) |$$
$$+ \cdots + N^0 | u_{yy}(x + h, y + k) - u_{yy}(x, y) |.$$

Hence

$$| \Pi(x, y) |_\mu \leqq N^0 + N_\mu + 6N^0 \| u \|_\mu. \tag{93}$$

If $u(x,y)$ and $u^*(x,y)$ form a pair of functions, of class $C''(\mu)$ on (86), satisfying

$$(94)\qquad \| u \|_\mu \leqq m \quad \text{and} \quad \| u^* \|_\mu \leqq m, \qquad (0 < m < 1),$$

and if $\Pi^*(x,y)$ denotes $\Pi(x,y,u^*(x,y),\cdots,u^*_{yy}(x,y))$, then there exist constants α, β, γ which are independent of δ and m (for small $\delta > 0$ and $m > 0$) and are such that

$$(95)\qquad | \Pi(x,y) - \Pi^*(x,y) |_\mu \leqq (\alpha N^0 + \beta N m + \gamma N_\lambda) \| u - u^* \|_\mu.$$

This can be proved by a device of Lichtenstein [15], pp. 93-94, as follows:

Let $\Phi(\tau) = \Phi(x,y,\zeta + \tau u,\cdots,\zeta_{yy} + \tau u_{yy})$ and let $\Phi^*(\tau)$ be defined analogously. Let $X(\tau) = \Phi(x_0,y_0,\zeta + \tau u,\cdots,\zeta_{yy} + \tau u_{yy})$ and let $X^*(\tau)$ be defined similarly. Then $-\Pi(x,y) = \Phi(1) - X'(0)$ and $-\Pi^*(x,y) = \Phi^*(1) - X^{*\prime}(0)$, where $' = d/d\tau$. Since $\Phi(0) - \Phi^*(0) = 0$,

$$(96)\qquad \Pi^* - \Pi = \int_0^1 \{(\Phi' - \Phi^{*\prime}) + X^{*\prime}(0) - X'(0)\} d\tau.$$

But the integrand in (96) can be written as

$$(97)\qquad (u - u^*)\{\Phi_z(x,y,\zeta + \tau u,\cdots) - A\} + u^*\{\Phi_z(x,y,\zeta + \tau u,\cdots) - \Phi_z(x,y,\zeta + \tau u^*,\cdots)\} + \cdots,$$

where the last three dots indicate five more pairs of terms analogous to the pair displayed explicitly, and $A, B, \cdots, F$ are the coefficients of (83). Since the coefficient of $u - u^*$ is the partial derivative of $-\Pi$ with respect to z at the point $(x,y,\tau u,\cdots,\tau u_{yy})$, the integrand of (96) has the majorant $6N^0 \| u - u^* \|_\mu + 36N \| u^* \|_\mu \| u - u^* \|_\mu$. Consequently, $| \Pi - \Pi^* |$ does not exceed a bound of the form occurring on the right-hand side of (95).

For any function $g = g(x,y)$, let $\Delta g = g(x+h, y+k) - g(x,y)$. Then, for fixed τ and $w = z, p, \cdots, s, t$,

$$(98)\qquad | \Delta\Phi_w(x,y,\zeta + \tau u,\cdots) | \leqq (N_\mu + 6N\tau \| u \|_\mu)(h^2 + k^2)^{\frac{1}{2}\mu};$$

cf. the derivation of (93). Also, for $h^2 + k^2 \neq 0$,

$$(99)\qquad (h^2 + k^2)^{-\frac{1}{2}\mu} | \Delta\{\Phi_w(x,y,\zeta + \tau u,\cdots) - \Phi_w(x,y,\zeta + \tau u^*,\cdots)\} |$$

does not exceed a bound of the form $(\alpha_1 N + \beta_1 N_\mu) \| u - u \|_\mu$, where α_1, β_1 are constants. In order to see this, let

$$\Phi^\tau(\sigma) = \Phi_w(x,y,\zeta + \tau u^* + \sigma\tau(u - u^*),\cdots).$$

Then $\Phi_w(x,y,\zeta + \tau u,\cdots) - \Phi_w(x,y,\zeta + \tau u^*,\cdots) = \Phi^\tau(1) - \Phi^\tau(0)$. But

$$d\Phi^\tau(\sigma)/d\sigma = \tau(u - u^*)\Phi_{wz} + (u_x - u^*_x)\Phi_{wp} + \cdots,$$

where the argument of $\Phi_{wz}, \Phi_{wp}, \cdots$ is $(x, y, \zeta + \tau u^* + \sigma\tau(u - u^*), \cdots)$. Since the third order partial derivatives of $\Phi(x, y, z, \cdots, t)$ with respect to $z, p, \cdots, t$ are bounded by N, it follows that $(h^2 + k^2)^{-\frac{1}{2}\mu}\Delta\{d\Phi^\tau(\sigma)/d\sigma\}$ has a bound of the type $(\alpha_2 N + \beta_2 N_\mu) \| u - u^* \|_\mu$ (cf. the derivation of (93)); hence the same holds for (99).

In view of (96), (97), (98) and the bound for (99), it is clear that there exist constants α, β, γ (independent of δ) such that (95) holds.

Theorem (§) can now be proved by the method of successive approximations. To this end, the following consequence of a classical result of Korn [14] will be needed: Let A, B, C, D, E, F be analytic functions of (x, y) on a set containing the circle (86), where $\delta > 0$ is sufficiently small, and let these functions satisfy

$$B^2 - AC < 0. \tag{100}$$

Let $v(x, y)$ be defined on the closure of (86) and satisfy there a uniform Hölder condition of order μ. Then

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} + Du_x + Eu_y + F = v \tag{101}$$

has a unique solution $u = u(x, y)$ which is of class $C''(\mu)$ on the closure of $\mathcal{C}_\delta$ and vanishes on the boundary of $\mathcal{C}_\delta$. Furthermore, there exists a constant M depending on μ but independent of δ (for small δ) such that

$$\| u \|_\mu \leqq M \mid v \mid_\mu, \qquad (M = M_\mu). \tag{102}$$

Let μ be any number satisfying $0 < \mu < \lambda$. Let A, B, C, D, E, F in (101) be identified with the coefficients of (83). With reference to the constant M in (102), let $\delta > 0$, $m > 0$ be fixed so small that

$$M(N^0 + N_\mu + 6Nm) < m \tag{103}$$

and

$$M(\alpha N^0 + \beta N m + \gamma N_\mu) < \epsilon < 1. \tag{104}$$

This is clearly possible in view of the remarks concerning (91), (92) and of the independence of the constants α, β, γ in (95) of δ and m.

Define on the closure of $\mathcal{C}_\delta$ a sequence of functions $u^0, u^1, \cdots$ of class $C''(\mu)$ by induction, as follows: Let $u^0 \equiv 0$. Suppose that $u^0, u^1, \cdots, u^n$ have been defined, are of class $C''(\mu)$ on the closure of $\mathcal{C}_\delta$ and satisfy

$$\| u^k \|_\mu \leqq m \tag{105}$$

for $k = 1, \cdots, n$. Then, by (93),

(106) $$|\Pi^n|_\mu \leqq N^0 + N_\mu + 6N^0 m,$$
where
(107) $$\Pi^n = \Pi^n(x, y) = \Pi(x, y, u^n, u^n_x, \cdots, u^n_{yy}).$$

Let $u = u^{n+1}$ be the unique solution, of class $C''(\mu)$, of

(108) $$Au_{xx} + \cdots + Fu = \Pi^n$$

which vanishes on the boundary of $\mathfrak{B}_\delta$. Then u^{n+1} satisfies $\| u^{n+1} \|_\mu \leqq M \mid \Pi^n \mid$, by (102). Hence, (106) and (103) show that (105) holds for $k = n + 1$; so that the induction is complete.

The difference $u = u^{n+1} - u^n$ satisfies the partial differential equation

(109) $$Au_{xx} + \cdots + Fu = \Pi^n - \Pi^{n-1},$$

is of class $C''(\mu)$ on $\mathfrak{B}_\delta$ and vanishes on the boundary of $\mathfrak{B}_\delta$. Hence, by (102), $\| u^{n+1} - u^n \|_\mu \leqq M \mid \Pi^n - \Pi^{n-1} \mid_\mu$. According to (95),

$$|\Pi^n - \Pi^{n-1}|_\mu \leqq (\alpha N^0 + \beta N m + \gamma N_\mu) \| u^n - u^{n-1} \|_\mu.$$

Hence, by (104),

(110) $$\| u^{n+1} - u^n \|_\mu \leqq \epsilon \| u^n - u^{n-1} \|_\mu \qquad \text{for } n = 1, 2, \cdots.$$

It follows from $\epsilon < 1$ and from the definition of the norm $\| \cdots \|_\mu$ that the series $u^1 + (u^2 - u^1) + \cdots$ is absolutely and uniformly convergent on $\mathfrak{B}_\delta$ to a function u of class $C''(\mu)$, and that this series can be differentiated term by term to obtain the first and second order partial derivatives of u.

Standard arguments show that $z = \zeta + u$ is a solution of (79) on $\mathfrak{B}_\delta$. Since $u^1 - u^0 = u^1$ satisfies $\| u^1 - u^0 \|_\mu \leqq m$, it follows from (110) that $\| u \|_\mu \leqq m/(1 - \epsilon)$, that is, that $\| z - \zeta \|_\mu \leqq m/(1 - \epsilon)$. In view of the fact that δ (hence m and ϵ) can be chosen arbitrarily small, and that $(x, y, \zeta, \zeta_x, \cdots, \zeta_{yy})$ reduces to $(x_0, y_0, z_0, p_0, \cdots, t_0)$ at $(x, y) = (x_0, y_0)$, the assertion (§) follows.

Part VII.

10. Let

(111) $$ds^2 = g_{ik}(u, v)\,du^i du^k, \qquad (u, v) = (u^1, u^2),$$

be a C'-metric on a vicinity D of $(u, v) = (0, 0)$ and let $\Gamma^j_{ik} = \Gamma^j_{ik}(u, v)$ be the corresponding Christoffel symbols. It is known ([3], p. 724) that given initial conditions need not determine a *unique* solution of

(112) $$u^{i\prime\prime} + \Gamma^i_{jk} u^{j\prime} u^{k\prime} = 0, \qquad i = 1, 2,$$

the differential equations of the geodesics of (111). The notion of a curvature of a C'-metric, described in Section **1** above, makes possible the formulation of the following theorem:

(I) *If* (111) *is a C'-metric on a (simply connected) vicinity D of $(u, v) = (0, 0)$ and possesses a curvature $K = K(u, v)$ which is a bounded function, then arbitrary initial conditions determine (locally) a unique solution of* (112).

This theorem is implied by the proof of theorem (I) in [3], p. 723. The assumptions of that theorem are quite different from those of (I) above; however, a perusal of the proof ([3], pp. 724-726) shows that it depends merely on the existence of a bounded curvature $K(u, v)$ in the sense of Section **1** above.

Under the assumptions of (I), there belongs to every ϕ a unique geodesic $J = J_\phi$, say

$$u = u(r, \phi), \quad v = v(r, \phi), \qquad (0 \leqq r \leqq \text{const.}, 0 \leqq \phi < 2\pi), \tag{113}$$

on which r is the arc-length,

$$u(0, \phi) = 0, \qquad v(0, \phi) = 0, \tag{114}$$

and ϕ is the angle between J_ϕ and a fixed direction at $(u, v) = (0, 0)$. The pair of functions (113) maps every sufficiently small circle $(0 \leqq r < \text{const.}, 0 \leqq \phi < 2\pi)$ in a continuous one-to-one manner on a vicinity of $(u, v) = (0, 0)$.

If the assumption of a *bounded* curvature for (111) is strengthened to the assumption of a *continuous* curvature, then the assertion of (I) can be strengthened as follows:

(II) *In addition to the assumptions of* (I), *let it be assumed that the curvature $K = K(u, v)$ of* (111) *is continuous. Let* (113) *be the unique geodesic satisfying* (114) *and making the angle ϕ with a fixed direction at $(u, v) = (0, 0)$, and let r be the arc-length on the geodesic* (113). *Then both functions* (113) *as well as their partial derivatives $u_r(r, \phi)$, $v_r(r, \phi)$ are of class C', and every sufficiently small circle $(0 \leqq r \leqq \text{const.}, 0 \leqq \phi < 2\pi)$ is mapped by* (113) *onto a vicinity of $(u, v) = (0, 0)$ in a one-to-one continuous manner and in such a way that, by virtue of this mapping,* (111) *becomes of the form*

$$ds^2 = dr^2 + g^2 d\phi^2, \tag{115}$$

where $g = g(r, \phi)$ is a continuous function possessing a continuous partial

derivative $g_r = g_r(r, \phi)$ *and satisfying*

(115 bis) $\qquad g(0, \phi) \equiv 0, \qquad g_r(0, \phi) \equiv 1$, and $g(r, \phi) > 0$ if $r > 0$.

This theorem is implied by the proof of theorem (III) of [9], p. 133, adapted from the proof of theorem 2 of [3], p. 724. It is easily seen that the proof of theorem (III) in [9], pp. 139-143, depends only on the existence of a *continuous* curvature, which proves (II) above.

It should be mentioned here that the wording of theorem (III) in [9] is erroneous; a corrected version is given by (II) above. The assumptions of theorem (III) in [9] imply only that (111) has a bounded curvature K, whereas its proof (cf. *loc. cit.*, the bottom of p. 141 and top of p. 142) requires somewhat more; for example, the continuity of K. In Section **12** below, there will be given an example showing that theorem (III) in [9] is false and that the assumption of (I) above (that is, the existence and the boundedness of K) do not imply the assertion of (II) above, claiming that (113) must be of class C'.

11. Under the assumptions of (II) above, it is possible to make an additional statement concerning the function g in (114), as follows:

LEMMA. *Under the assumptions of* (II), *the function* $g = g(r, \phi)$ *in* (114) *has a continuous second partial derivative* $g_{rr}(r, \phi)$ *with respect to* r *and satisfies the Jacobi equation*

$$g_{rr} + Kg = 0, \qquad r > 0, \tag{116}$$

where $K = K(u, v)$ *is considered as a function of* (r, ϕ) *by virtue of* (113).

Essentially, this lemma was announced in [3], where a proof is indicated. In view of the difficulties involved in the details, a complete verification of the Lemma will be given here.

For a fixed (small) $r = c > 0$, (113) is a Jordan curve J^c of class C'. In fact, it is of class C'', although (113) need not be a C''-parametrization of J^r (cf. [4], or the example in Section **12** below). In order to show that J^c is of class C'', consider the inverse of the transformation (113),

$$r = r(u, v), \qquad \phi = \phi(u, v). \tag{117}$$

The transformation (117) is of class C' and has a non-vanishing Jacobian in a punctured vicinity ($0 < u^2 + v^2 < \text{const.}$) of $(u, v) = (0, 0)$. The transformation rule of the contravariant form of the tensors occurring in (111)

and (115) implies, on this punctured vicinity, the identity

$$(118) \qquad g^{ik} r_i r_k = 1, \qquad (u^2 + v^2 > 0),$$

where $(g^{ik}) = (g_{ik})^{-1}$ and $r_1 = \partial r/\partial u$, $r_2 = \partial r/\partial v$. Locally, the partial differential equation (118) for $r = r(u, v)$ can be written in the form

$$(118\text{ bis}) \qquad r_u = F(r_v; u, v),$$

where F is analytic in r_v for fixed (u, v), and F and its partial derivatives with respect to r_v are of class C' in $(r_v; u, v)$.

Consider a point $(r, \phi) = (c, \phi^0)$ on the arc J^c. It can be supposed that the geodesic arc $\phi = \phi^0$ in (113) is the arc $u = 0$, for small $v \geqq 0$. In fact, the geodesic arc consisting of the geodesics $\phi = \phi^0$ and $\phi = \phi^0 + \pi$ in (113) has, for small $|v|$, a parametrization of the form $u = u(v)$, and the change of parameters $(u, v) \rightarrow (u - u(v), \pm v)$ is of class C'', has a non-vanishing Jacobian, leaves the assumptions of (II) invariant and transforms the arc consisting of $\phi = \phi^0$ and $\phi = \phi^0 + \pi$ into $u = 0$.

On the geodesic $u = 0$, the arc-length $r = r(0, v)$ is of class C'' as a function of v. The standard existence and uniqueness theorems for the partial differential equation of type (118 bis) show that $r = r(u, v)$ is of class C'' in a vicinity of the point $(u, v) = (0, v^0)$, corresponding to $(r, \phi) = (c, \phi^0)$. Since $r_u{}^2 + r_v{}^2 \neq 0$ at $(u, v) = (0, v^0) \neq (0, 0)$, it follows that the neighborhood of $(0, v^0)$ on the curve J^c: $r(u, v) = c$ is an arc of class C'', and so, since (c, ϕ^0) is an arbitrary point on the Jordan arc J^c, the latter is a curve of class C''.

Since (111) has a curvature K, the formula of Gauss-Bonnet,

$$(119) \qquad \int_J \kappa ds + \Sigma\, a_k + \iint_E K(g_{11}g_{22} - g_{12}{}^2)^{\frac{1}{2}}\, dudv = 2\pi,$$

is applicable to every region E bounded by a Jordan curve J which is piecewise of class C'' (cf. Section **1** above). Hence, an admissible domain $E = E(J)$ is

$$(120) \qquad E\colon\ (0 <)\ r_1 \leqq r \leqq r_2, \qquad \phi_1 \leqq \phi \leqq \phi_2$$

if r_2 is sufficiently small and $\phi_2 - \phi_1 < 2\pi$. In fact, the boundary J of (120) consists of two geodesic arcs, $\phi = \phi_1$ and $\phi = \phi_2$, where $r_1 \leqq r \leqq r_2$ (hence $r \neq 0$) and of arcs of two orthogonal trajectories J^c, where $c = r_1, r_2$. Clearly, $\Sigma a_k = 2\pi$ in the case (120) of (119). Hence the change of the integration variables u, v to r, ϕ in the case (120) of (119) gives

$$\int_J \kappa g ds = \int\int_E K g dr d\phi, \tag{121}$$

since $ds = g d\phi$ and $g^2 = (g_{11}g_{22} - g_{12})^2(\partial(u,v)/\partial(r,\phi))^2$. On letting ϕ_1 and ϕ_2 tend to ϕ, and then r_1 and r_2 to r (>0), it follows from the continuity of K and g that $\kappa = \kappa(r,\phi)$, the geodesic curvature of J^r at (r,ϕ), has with respect to r a partial derivative satisfying

$$(\kappa g)_r + Kg = 0. \tag{122}$$

Hence, in order to prove the Lemma, it remains only to show that the geodesic curvature κ satisfies

$$\kappa g = g_r. \tag{123}$$

Strictly speaking, κ in (119) is defined (on smooth sub-arcs of J) as $d\theta/ds$, where $-\theta$ is a (continuous) angle from a vector tangent to J to a vector in a field which is parallel in the sense of Levi-Civita along J; on the other hand, the geodesic curvature is $|\,d\theta/ds\,|$.

If (113) represents a C''-parametrization of J^c, so that g in (115) is of class C', then (123) follows from standard formulae for $d\theta/ds$ (cf., e. g., [1], p. 267). The point in the relation (123) is that (123) holds despite the fact that g need not be of class C'.

In order to prove (123) at a point $(r,\phi) = (c,\phi^0)$, where $c > 0$ is small, it can be supposed that $\phi^0 = 0$. Put

$$\psi = \psi(\phi) = \int_0^\phi g(c,t)dt \tag{124}$$

for ϕ near 0. Then ψ is of class C' (since g is continuous) and $\psi_\phi = g \neq 0$. Hence the function (124) has an inverse of class C',

$$\phi = \phi(\psi), \tag{125}$$

for ψ near 0. Consider the transformation $(u,v) \rightarrow (r,\psi)$,

$$u = u(r;\psi), \qquad v = r(r;\psi) \tag{126}$$

which results by substituting (125) into (113). Then (111) or (115) is transformed into

$$ds^2 = dr^2 + G d\psi^2, \quad \text{where} \quad G = g^2(r,\phi)/g^2(c,\phi) \tag{127}$$

is a continuous function of (r,ψ) near $(r,\psi) = (c,0)$. The function G (as well as the other coefficients 1, 0 in (127)) can be calculated from the tensor transformation rule

(128) $$G = g_{ik}u^i{}_\psi u^k{}_\psi, \quad \text{where} \quad u^i = u^i(r;\psi),$$

and $u^1{}_\psi = \partial u/\partial\psi$, $u^2{}_\psi = \partial v/\partial\psi$ as well as the partial derivatives $u^i{}_r$, $u^i{}_\psi$, $u^i{}_{rr}$, $u^i{}_{r\psi} = u^i{}_{\psi r}$ exist and are continuous, by (II). In addition, $u^i{}_{\psi\psi}(c,\psi)$ exists and is continuous, since (124) shows that $r = c$ in (126) gives an arc-length parametrization of a portion of the curve J^c, which is of class C''. It follows that $G_r(r,\psi)$ and $G_\psi(c,\psi)$ exist and are continuous; moreover, these partial derivatives can be calculated from (128) by formal rules. Thus, although (127) is not a C'-metric, it has Christoffel symbols, say $\gamma^j{}_{ik}$, at the points (c,ψ), for small $|\psi|$, and these $\gamma^j{}_{ik}$ can be calculated in terms of the Christoffel symbols $\Gamma^j{}_{ik}$ of (111) by the standard transformation rule,

(129) $$\gamma^a{}_{jk}u^i{}_a = \Gamma^i{}_{a\beta}u^a{}_j u^\beta{}_k + u^i{}_{jk},$$

where $u^1 = u$, $u^2 = v$ are given by (126), the subscripts on u^i denote partial differentiation with respect to $v^2 = r$, $v^2 = \psi$, and the arguments of all functions correspond to $(r,\psi) = (c,\psi)$.

With reference to the parameters $u^1 = u$, $u^2 = v$, the differential equations defining parallel transport along J^c are

(130) $$W^{i\prime} + \Gamma^i{}_{jk}W^j u^{k\prime} = 0, \quad \text{where} \quad {}^\prime = d/d\psi; \quad i = 1, 2.$$

In view of (129), this reduces to

(131) $$w^{i\prime} + \gamma^i{}_{jk}w^j v^{k\prime} = 0$$

by virtue of

(132) $$w^i = W^a v^i{}_a \quad \text{and/or} \quad W^i = w^a u^i{}_a.$$

In order to see this, note that $W^{i\prime} = w^{a\prime}u^i{}_a + wu^i{}_{a\beta}v^{\beta\prime}$ and that the second term of (130) is $\Gamma^i{}_{jk}w^a u^j{}_a u^k{}_\beta v^{\beta\prime}$; hence (131) follows from (129) if (130) is multiplied by $v^n{}_i$. Since $v^{1\prime} = 0$, $v^{2\prime} = 1$, the pair of equations (131) is $w^{i\prime} + \gamma^i{}_{j2}w^j = 0$. In view of $G(c,\psi) = 1$ (hence $G_\psi(c,\psi) = 0$), it follows that

$$\gamma^1{}_{12} = 0, \qquad \gamma^1{}_{22} = -\tfrac{1}{2}G_r(c,\psi) = -g_r(c,\phi)/g(c,\phi),$$
$$\gamma^2{}_{12} = \tfrac{1}{2}G_r = g_r(c,\phi)/g(c,\phi), \qquad \gamma^2{}_{22} = 0;$$

so that (131) becomes

(133) $$w^{1\prime} - (g_r/g)w^2 = 0, \qquad w^{2\prime} + (g_r/g)w^1 = 0.$$

In (123), $\kappa = d\theta/ds$, where s is ψ and $\theta = \theta(s)$ is the angle (in the metric (111)) from any solution vector $(W^1(s), W^2(s)) \neq 0$ of (130) to the vector (u_ψ, v_ψ) tangent to J^c. Since length and angles are preserved under

transformations of class C' (on vectors and metrics), it follows from (132) that θ can be interpreted as the angle from any solution vector $(w^1(s), w^2(s)) \neq 0$ of (133) to the vector $(r_\psi, \psi_\psi) = (0, 1)$ tangent to J^c. Consider the solution of (133) satisfying the initial condition $w^1(0) = 0$, $w^2(0) = 1$. Then, if $\theta = \theta(s)$ is the angle from this solution vector to $(0, 1)$, it follows that $\sin \theta = w^1$, hence $d\theta/ds = w^{1\prime}$ at $s = 0$. Thus, the first equation in (133) and $w^2(0) = 1$ imply (123) at the point $(c, \phi^0) = (c, 0)$ of J^c. This proves the Lemma.

Since g in (115) need not be of class C' under the assumptions of (II), is is not possible to consider the differential equations of the geodesics in the (r, ϕ)-parameters. This defect in (II) can be remedied by an additional hypothesis on K, as follows:

(III) *In addtion to the assumptions of* (II), *let it be assumed that the curvature $K = K(u, v)$ of* (111) *is of class C'. Then the functions* (113) *are of class C'', and so the function g in* (115) *is of class C'.*

Assuming the Lemma above, an analogous theorem was proved in [5], p. 223, as follows: The assumption on K and the fact that (113) is of class C' imply that K in (116) is of class C' as a function of (r, ϕ). Hence, that solution $g = g(r, \phi)$ of the ordinary differential equation (116) which is deterimned by the initial conditions $g(0, \phi) = 0$, $g_r(0, \phi) = 1$ is of class C', and so (115) is a C'-metric for $r > 0$. It follows from the theorem (**) in [5], p. 222, that the functions occurring in (113) are of class C'' for small $u^2 + v^2 > 0$ (where $\partial(u, v)/\partial(r, \phi) \neq 0$). The difficulty at $(u, v) = (0, 0)$ can be eliminated by first considering u, v as functions, not of (r, ϕ), but of the Riemann coordinates $x = r \cos \phi$, $y = r \sin \phi$, and then verifying that (111) is transformed into a C'-metric in (dx, dy).

Clearly, (i) of Section **2** above can be deduced from (III).

(II_n)-(III_n) *Let* (111) *be a C^n-metric possessing a curvature $K = K(u, v)$ of class C^{n-1} or C^n, where $n \geqq 1$. Then the functions* (113) *are of class C^n, C^{n+1}, respectively, and g in* (115) *is correspondingly of class C^{n-1} or C^n.*

These theorems are proved in the same way as their particular cases, (II) and (III). (Actually, the cases $n \geqq 2$ are quite simple when compared to the cases $n = 1$).

12. There will now be given an example of a C'-metric (111) for which the partial derivatives $\partial g_{ik}/\partial u^j$ satisfy a uniform Lipschitz condition (which by [9], pp. 139-140, implies that there exists a bounded curvature

$K = K(u, v)$), but the corresponding functions (113) are not of class C'.

Such a metric is, for instance,

$$(134) \qquad ds^2 = h(v)(du^2 + dv^2), \text{ where } h(v) = 1 + \tfrac{1}{2}v^2 \operatorname{sgn} v,$$

if the domain D is the strip $|u| < \infty$, $|v| < 2^{\frac{1}{2}}$. If $v \neq 0$, it is readily calculated that (134) has the curvature

$$K = -\tfrac{1}{2}h^{-1}(\log h)_{vv} = -\tfrac{1}{2}(1 + \tfrac{1}{2}v^2 \operatorname{sgn} v)^{-3}(\operatorname{sgn} v - \tfrac{1}{2}v^2).$$

It follows that (134) has on D a curvature $K(u, v)$ which is bounded but not continuous ($K(u, +0) = \frac{1}{2}$, $K(u, -0) = \frac{1}{2}$). If u is used as a parameter, then (112) becomes

$$(135) \quad 2v''/(1 + v'^2) = (\log h)_v = |v|/(1 + \tfrac{1}{2}v^2 \operatorname{sgn} v), \text{ where } {}' = d/du;$$

cf. [1], p. 277. It is easily verified that the solution $v = v(u) = v(u; \phi)$ of (135) satisfying $v(0) = 0$ and $dv(0)/du = \tan\phi$ is given by

$$(136_1) \qquad v = 2^{\frac{1}{2}} \sinh(2^{-\frac{1}{2}}u/\cos\phi)\sin\phi \text{ for } u \geqq 0 \text{ when } 0 \leqq \phi < \tfrac{1}{2}\pi,$$

$$(136_2) \qquad v = 2^{\frac{1}{2}} \sin(2^{-\frac{1}{2}}u/\cos\phi)\sin\phi \text{ for } u \geqq 0 \text{ when } -\tfrac{1}{2}\pi < \phi < 0.$$

Hence $v(u; \phi)$ is continuous for $u \geqq 0$, $|\phi| < \frac{1}{2}\pi$. It is clear that $v(u; \phi)$ has a continuous partial derivative v_ϕ for $\phi \neq 0$ and that this derivative has the limits

$$(137_1) \qquad v_\phi(u; +0) = 2^{\frac{1}{2}} \sinh(2^{-\frac{1}{2}}u) \text{ for } u \geqq 0,$$

$$(137_2) \qquad v_\phi(u; -0) = 2^{\frac{1}{2}} \sin(2^{-\frac{1}{2}}u) \text{ for } u \geqq 0.$$

Thus, $v(u; \phi)$ is not of class C' (in fact, $v_\phi(u; 0)$ does not exist for small $u > 0$). It will be shown that, for this reason, the function $v = v(r, \phi)$ occurring in (113) cannot be class C'.

According to (134), the arc-length $r = r(u; \phi)$ on the geodesic (136_1) or (136_2) satisfies the relation $dr = h^{\frac{1}{2}}(v)(1 + (dv/du)^2)^{\frac{1}{2}}du$. Hence (136_1), (136_2) imply that

$$(138_1) \qquad r = (\cos\phi)^{-1} \int_0^u \{1 + \sinh^2(2^{-\frac{1}{2}}\tau/\cos\phi)\sin^2\phi\} d\tau,$$

$$(138_2) \qquad r = (\cos\phi)^{-1} \int_0^u \{1 + \sin^2(2^{-\frac{1}{2}}\tau/\cos\phi)\sin^2\phi\} d\tau,$$

respectively. It follows that $r(u; \phi)$ has a continuous partial derivative r_ϕ for $u \geqq 0$, $|\phi| < \frac{1}{2}\pi$. This is clear for $\phi > 0$ and $\phi < 0$ from (138_1) and

(138_2), respectively. If $\phi = 0$, it is sufficient to note that ϕ occurs in (138_1), (138_2) only in $\cos\phi$ and $\sin^2\phi$; so that $r_\phi(u;0)$ exists and is 0. Furthermore it is easily seen that $r_\phi(u;\phi)$ is continuous at $\phi = 0$ also. Hence, $r(u;\phi)$ is of class C'. Since $r_u \neq 0$, it follows that $r = r(u;\phi)$ can be solved for u and gives the function $u = u(r,\phi)$ in (113). Hence $u = u(r,\phi)$ is of class C'. The function $v = v(r,\phi)$ in (113) results by substituting $u = u(r,\phi)$ into $v = v(r,\phi)$. Thus, for $\phi \neq 0$, the function $v(r,\phi)$ has a continuous partial derivative,

$$v_\phi(r,\phi) = v_u(u;\phi)u_\phi(r,\phi) + v_\phi(u;\phi) \tag{139}$$

Since $v_u(u;\phi)$ is $\cosh(2^{-\frac{1}{2}}u/\cos\phi)\tan\phi$ or $\cos(2^{-\frac{1}{2}}u/\cos\phi)\tan\phi$ according as $\phi > 0$ or $\phi < 0$, it follows that $v_u(u;\phi) \to 0$ as $\phi \to 0$. But the continuity of $u_\phi(r,\phi)$ shows that $v_\phi(r,\phi)$ has the limits

$$v_\phi(r,+0) = v_\phi(u;+0) \text{ and } v_\phi(r,-0) = v_\phi(u;-0), \quad \text{where } u = u(r,\phi). \tag{140}$$

Hence it is seen from (137_1), (137_2) that the function $v(r,\phi)$ occurring in (113) is not of class C' (in fact, it has at $\phi = 0$ no partial derivative with respect to ϕ, for small $r > 0$.

Appendix.*

The object of this Appendix is to prove the following uniqueness theorem on closed convex surfaces:

($\square$) *Let $F(U, V; \lambda, \mu, \nu)$ be defined and of class C'' on the four-dimensional set*

$$U > 0, U^2 \geqq 4V > 0, \qquad \lambda^2 + \mu^2 + \nu^2 = 1$$

and satisfy

$$\operatorname{sgn} \partial F/\partial R_1 = \operatorname{sgn} \partial F/\partial R_2 \neq 0, \qquad U = R_1 + R_2, V = R_1R_2. \tag{1}$$

Then (up to translations) there is at most one closed surface S of class C''' with positive Gaussian curvature such that the principal radii of curvature R_1, R_2 at the point of S, where the inward unit normal vector is (λ, μ, ν), satisfies

$$F(R_1 + R_2, R_1R_2; \lambda, \mu, \nu) = 0. \tag{2}$$

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* Added January 21, 1953.

When F is of the form $R_1 + R_2 - \phi(\lambda, \mu, \nu)$ or $R_1R_2 - \phi(\lambda, \mu, \nu)$, then ($\square$) reduces to classical theorems (Christoffel, Minkowski), which are known to be valid under lighter smoothness assumptions on S, $\phi(\lambda, \mu, \nu)$ than what is assumed by ($\square$).

The assertion ($\square$) is due to Alexandroff [*1*] if F and S are restricted to be analytic. It has been proved by Pogoreloff [*5*] under the lighter hypothesis that F and S are of class C''' and C'''' (instead of C'' and C''', as in the above theorem), respectively. While the version of the theorem to be proved reduces by *one* the degree of differentiability assumed by Pogoreloff, it would be desirable to reduce the assumption of differentiability on S from C''' to C''; the class C'' being the natural one, the class of lowest differentiability in which R_1, R_2 are defined (by standard formulae) and are continuous. But the possibility of this reduction, $C''' \rightarrow C''$, will be left undecided.

By the assumption that F is of class C'' is meant that if, near any point of the sphere $\lambda^2 + \mu^2 + \nu^2 = 1$, one of the variables λ, μ, ν is expressed as a function of the other two (for example, $\nu = (1 - \lambda^2 - \mu^2)^{\frac{1}{2}} > 0$), then F is a function of class C'' of the four variables $(U, V; \lambda, \mu)$. Condition (*1*) means that the symbol $\neq$ in (74) can be replaced by $>$ and so the differential equation (76) is of elliptic type.

By using Theorem (†) in [*3*] (for a non-analytic elliptic equation), instead of the Lemma of Lewy [*4*], pp. 259-260 (for analytic elliptic equations), it is possible to prove ($\square$) by the procedure which was used by Lewy [*4*], pp. 261-262, to prove the analytic case of the theorem of Minkowski ($F = R_1R_2 - \phi$) and which was adapted by Lewy from arguments applied by Cohn-Vossen [*2*], pp. 125-132, in an analogous problem.

Proof of ($\square$). Suppose that S and Σ are two surfaces (in an (x, y, z)-space) satisfying the conditions of ($\square$). It must be shown that Σ is a translation of S. According to (ii) and (ii bis) above, in a neighborhood of a point of the unit sphere, say of the point $(\lambda, \mu, \nu) = (0, 0, 1)$, the surfaces S and Σ, being of class C''', possess spherical C''-parametrizations $X(\lambda, \mu)$ and $\Xi(\lambda, \mu)$, respectively; so that (λ, μ, ν), where $\nu = (1 - \lambda^2 - \mu^2)^{\frac{1}{2}} > 0$, is the inward normal vector at the point $X(\lambda, \mu)$, $\Xi(\lambda, \mu)$ of S, Σ. Let $(h_{ik}(\lambda, \mu))$ and $(\eta_{ik}(\lambda, \mu))$ be the matrices of the second fundamental forms of S and Σ in terms of the parameters $(\lambda, \mu) = (\lambda^1, \lambda^2)$.

Consider, with Maxwell, those points (λ, μ), called by Cohn-Vossen "congruence points," at which $(h_{ik}) = (\eta_{ik})$. Then (*1*), (*2*) imply that either (λ, μ) is a congruence point or $\det(h_{ik} - \eta_{ik}) < 0$ at the point (λ, μ).

In order to see this, note that, by (33), the principal curvatures R_1, R_2 of S and Σ are the characteristic numbers of the matrix products $(f_{ik})^{-1}(h_{ik})$ and $(f_{ik})^{-1}(\eta_{ik})$, respectively, where (f_{ik}) is the common third fundamental matrix (58) of S and Σ. It is convenient to replace the matrix product $(f_{ik})^{-1}(h_{ik})$ by the positive definite symmetric matrix $(t_{ik}) = (f_{ik})^{-\frac{1}{2}}(h_{ik})(f_{ik})^{-\frac{1}{2}}$, having the same characteristic numbers, where $(f_{ik})^{\frac{1}{2}}$ is the (unique) positive definite square root of (f_{ik}). Similarly, $(f_{ik})^{-1}(\eta_{ik})$ can be replaced by $(\tau_{ik}) = (f_{ik})^{-\frac{1}{2}}(\eta_{ik})(f_{ik})^{-\frac{1}{2}}$. If R_1, R_2 $(\geqq R_1)$ are the characteristic numbers of (t_{ik}) and ρ_1, ρ_2 $(\geqq \rho_1)$ are those of (τ_{ik}), then (*1*) and (*2*) show that, unless $R_1 = \rho_1$ and $R_2 = \rho_2$, either $R_1 < \rho_1$ and $R_2 > \rho_2$ or $R_1 > \rho_1$ and $R_2 < \rho_2$. Hence, unless $(t_{ik}) = (\tau_{ik})$, it follows that $\det(t_{ik} - \tau_{ik}) < 0$ (as can be seen by considering the ellipses $t_{ik}d\lambda^i d\lambda^k = 1$ and $\tau_{ik}d\lambda^i d\lambda^k = 1$ in the $(d\lambda^1, d\lambda^2)$-plane, since these ellipses either coincide or intersect in four points; cf. [*2*], pp. 125-126, or [*4*], p. 262). This implies the disjunctive alternative: Either $(h_{ik}) = (\eta_{ik})$ or $\det(h_{ik} - \eta_{ik}) < 0$.

Accordingly, the (λ, μ)-net defined by

$$(h_{ik} - \eta_{ik})\, d\lambda^i d\lambda^k = 0, \tag{3}$$

where $(\lambda^1, \lambda^2) = (\lambda, \mu)$, is real and has the congruence points as its singularities. It follows therefore by Cohn-Vossen's arguments in [*2*], pp. 126-127, that the uniqueness assertion of ($\square$) will be proved if it is shown that, unless S and Σ coincide after a translation, the congruence points are isolated and the index of any singular point of the net (*3*) is negative; cf. Lewy [*4*], pp. 261-262.

Let $w = X \cdot N$ and $\omega = \Xi \cdot N$ be the supporting functions (53) of S and Σ, respectively. Then $w(\lambda, \mu)$ and $\omega(\lambda, \mu)$ are of class C''', by (iv$_3$) above. Both w and ω satisfy the elliptic partial differential equation (*2*), where $R_1 + R_2 = 2H/K$ and $R_1 R_2 = 1/K$ are given by (65), (66) and $\nu = (1 - \lambda^2 - \mu^2)^{\frac{1}{2}} > 0$. If $(\lambda, \mu, \nu) = (0, 0, 1)$ is a congruence point, it can be supposed that Σ has been translated so that $X(0,0) = \Xi(0,0)$. Then the difference $w - \omega$ and its partial derivatives $w_\lambda - \omega_\lambda$, $w_\mu - \omega_\mu$ are 0 at $(\lambda, \mu) = (0, 0)$. Theorem (†) in [*3*] implies that, unless S and Σ coincide, there exist an integer $n > 1$ and a negative constant c such that if $\chi = (\lambda^2 + \mu^2)^{\frac{1}{2}}$ and $\chi \to 0$,

$$w - \omega = o(\chi^n), \quad w_{ik} - \omega_{ik} = O(\chi^{n-1}) \quad \text{and} \quad \det(w_{ik} - \omega_{ik}) \sim c\chi^{2n-2},$$

where the subscripts of w and ω denote partial differentiations with respect to $\lambda^1 = \lambda$, $\lambda^2 = \mu$.

If the last formula line is compared with (66), it is seen that

$$\det(h_{ik} - \eta_{ik}) \sim c\chi^{2n-2} \text{ as } \chi \to 0 \qquad (\chi^2 = \lambda^2 + \mu^2).$$

Since $c < 0$, the proof of the uniqueness theorem ($\square$) is now complete; cf. Lewy [4], p. 262.

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GENERALIZED ASYMPTOTIC DENSITY.*

By R. Creighton Buck.

1. Introduction. In this paper, we shall discuss the properties of an asymptotic density in the context of a general σ-finite measure space; in particular, we shall prove what may be called the additivity theorem. This was originally formulated and proved only in the very special case of asymptotic density of sets of integers [2]. This introduces a certain amount of unification into the theory of asymptotic density. In Section **3**, a number of applications will be made of the additivity theorem.

Let X be a set on which a density function is to be defined. Topological properties of X will play no role; however, the notion of boundedness is important. We suppose that a countable sequence of sets $K(1) \subset K(2) \subset \cdots$ has been chosen so that $\bigcup K(n) = X$. A set $S \subset X$ is called bounded if $S \subset K(n)$ for some n. Let μ_n be a sequence of measures on X, having in common a field $\mathfrak{M}$ of measurable sets which contains at least the sets $K(n)$ and X. We require two conditions: (i) $\mu_n(X) = 1$ for $n = 1, 2, \cdots$; (ii) $\mu_n(K(j)) \to 0$, as $n \to \infty$, for each $j = 1, 2, \cdots$. In all that follows, we shall be discussing sets in the class $\mathfrak{M}$. For certain of these, we define a density function $D(S)$ as $\lim \mu_n(S)$, when this limit exists. For any set S in $\mathfrak{M}$, upper and lower densities $\bar{D}(S)$ and $\underline{D}(S)$ are defined as the upper and lower limits of the sequence $\mu_n(S)$. Thus, S has density only when $\bar{D}(S)$ and $\underline{D}(S)$ agree.

Examples of densities are easily given. If ν is a measure on X for which $\nu(X) = +\infty$ while $\nu(K(j)) < \infty$ for all j, then ν defines a density on X with measures μ_n defined by $\mu_n(S) = \nu(S \cap K(n))/\nu(K(n))$. In particular, the customary asymptotic density defined for sets of integers is obtained by the choice of X as $\{1, 2, \cdots\}$ $K(n)$ as $\{1, 2, \cdots, n\}$ and ν as the point measure with mass 1 at each point.

The density function D with measures μ_n has many properties similar to that of a measure itself. These are summarized in the following statement:

1) $D(X) = 1$, and if S is bounded, $D(S) = 0$.

* Received November 10, 1952.

2) If S has density, so does $X - S$ and $D(X - S) = 1 - D(S)$.

3) If A and B have density, and $A \subset B$, then $D(A) \leqq D(B)$.

4) If A and B have density, and are disjoint, then $A \cup B$ has density, and $D(A \cup B) = D(A) + D(B)$.

However, in contrast, in 4) if A and B are not disjoint, then neither $A \cup B$ nor $A \cap B$ need have density. Moreover, D is not countably additive; for example, setting $J_n = K(n) - K(n-1)$ we have $D(J_n) = 0$ for all n while $D(\bigcup J_n) = D(X) = 1$. It is possible to overcome the latter defect by introducing a modified set inclusion and set union and thus obtain a true additivity theorem. The device is essentially that of working modulo bounded sets. We shall write $A \mathrel{\dot{\subset}} B$ when $A - B$ is bounded; there is then a value of j for which $A - K(j) \subset B - K(j)$. If $A \mathrel{\dot{\subset}} B$, then since bounded sets are of zero density, $\underline{D}(A) \leqq \underline{D}(B)$ and $\bar{D}(A) \leqq \bar{D}(B)$; thus, D is still monotone.

At this point we impose a further condition on the measures μ_n which is needed in the proof of the next theorem:

(iii) the measure μ_n has bounded support.

This requires that there exist a sequence $a(n)$ such that $\mu_n(S) = 0$ for any set S disjoint from $K(a(n))$. We observe that the density obtained from a single measure ν as described above obeys (iii); the support of μ_n is contained in $K(n)$ itself.

THEOREM 1.1. *Let* $A_1 \mathrel{\dot{\subset}} A_2 \mathrel{\dot{\subset}} A_3 \mathrel{\dot{\subset}} \cdots$ *and let* $\lim \bar{D}(A_n) = \Delta$ *and* $\lim \underline{D}(A_n) = \delta$. *Then, there exists a set A with* $\bar{D}(A) = \Delta$ *and* $\underline{D}(A) = \delta$, *such that* $A_n \mathrel{\dot{\subset}} A$ *for all* n.

COROLLARY 1. *If the sets A_n have density, and* $A_1 \mathrel{\dot{\subset}} A_2 \mathrel{\dot{\subset}} \cdots$ *then there is a set A, unique up to sets of zero density, such that* $A_n \mathrel{\dot{\subset}} A$ *for all n, and* $D(A) = \lim D(A_n)$.

COROLLARY 2. *If* $C_1, C_2, \cdots$ *are disjoint sets having density, there is a set C, unique up to sets of zero density, such that* $C \mathrel{\dot{\supset}} \bigcup_1^n C_k$ *for all n, and with* $D(C) = \sum_1^\infty D(C_k)$.

Proof of the theorem. Assume for the present that the sets A_n are monotone in the usual sense. Given $\epsilon > 0$, choose b_n so that $\mu_{b_n}(A_n) \leqq \underline{D}(A_n) + \epsilon$ and for all $k \geqq b_n$, $\mu_k(A_n) \leqq \bar{D}(A_n) + \epsilon$. With the sequence $a(n)$ having the meaning given above, let $I_1 = K(a(b_1))$, and $I_n = K(a(b_n)) - K(a(b_{n-1}))$.

The set A is defined by $A = \bigcup_{1}^{\infty} A_n \cap I_n$. Since $A_n \supset A_j$ for all $n \geqq j$, $A \cap \bigcup_{n\geqq j} I_n = \bigcup_{n\geqq j} A_n \cap I_n \supset \bigcap_{n\geqq j} A_j \cap I_n = A_j \cap \bigcup_{n\geqq j} I_n$ and $A \dot{\supset} A_j$. From the monotone character of upper and lower density, and the definition of Δ and δ, $\bar{D}(A) \geqq \Delta$ and $\underline{D}(A) \geqq \delta$. If $n \leqq j$, then $A_n \subset A_j$ so that

$$A \cap \bigcup_{n\leqq j} I_n = \bigcup_{n\leqq j} A_n \cap I_n \subset \bigcup A_j \cap I_n = A_j \cap \bigcup_{n\leqq j} I_n,$$

or $A \cap K(a(b_j)) \subset A_j \cap K(a(b_j))$. Since the support of μ_k lies in $K(a(k))$, $\mu_k(S) = \mu_k(S \cap K(a(b_j)))$ whenever $k \geqq b_j$. In particular, $\mu_k(A) \leqq \mu_k(A_j)$ when $k \geqq b_j$. Using the definition of b_j, we have $\mu_k(A) \leqq \bar{D}(A_j) + \epsilon$ for all $k \geqq b_j$, and $\mu_{b_j}(A) \leqq \underline{D}(A_j) + \epsilon$. This immediately yields $\bar{D}(A) \leqq \Delta + \epsilon$, and $\underline{D}(A) \leqq \delta + \epsilon$. Letting ϵ decrease, we see that $\bar{D}(A)$ and $\underline{D}(A)$ must equal Δ and δ respectively. Returning to the original hypothesis that $A_1 \dot{\subset} A_2 \dot{\subset} A_3 \dot{\subset} \cdots$, there exist bounded sets B_n such that $A_n - B_n \subset A_{n+1} - B_n$ for each n. Set $A_n^* = A_n \cup (B_1 \cup B_2 \cup \cdots \cup B_{n-1})$. Applying the previous argument to this sequence gives the desired result.

The only unproved statement in Corollary 1 is the uniqueness of A. If A^* is another set for which both $D(A^*) = \lim D(A_n)$ and $A_n \dot{\subset} A^*$ for all n, then so is the set $A \cap A^*$, and since $\mu_k(A) = \mu_k(A - A^*) + \mu_k(A \cap A^*)$, $D(A - A^*) = 0$. Likewise $D(A^* - A) = 0$, and A and A^* differ only by a set of zero density.

In connection with this theorem, there is a simpler result concerned with a single measure on X which, especially in its applications, has a certain similarity.

THEOREM 1.2. *Let ν be a measure on X, finite on bounded sets. Let $A_1 \subset A_2 \subset \cdots$ be a sequence of sets of finite measure. Then given $\delta > 0$, there is a set A with $\nu(A) < \delta$ and $A_n \dot{\subset} A$ for all n.*

Let $R_n = X - K(n)$. If B is of finite measure, then $\nu(B \cap R_n) \to 0$ as n increases. Let k_n be a value of k for which $\nu(A_n \cap R_k) < \delta/2^n$, and set $I_1 = K(k_1)$, $I_n = K(k_n) - K(k_{n-1})$. As before let $A = \bigcup A_n \cap I_n$; we again have $A \dot{\supset} A_j$ for all j. Computing $\nu(A)$, we have

$$\nu(A) \leqq \Sigma\nu(A_n \cap I_n) \leqq \Sigma\nu(A_n \cap R_{k_n}) \leqq \Sigma\delta/2^n = \delta.$$

The set A cannot be taken of zero measure unless each of the sets A_n is essentially bounded, in the sense that there is a bounded set B with $\nu(A - B) = 0$.

2. Applications. We shall be chiefly concerned with real valued functions defined on X, measurable, and bounded on bounded sets. In Section **4**, we shall discuss the extension to complex-valued functions. Although X has no topology, we can use the sets $R_n = X - K(n)$ to define one 'at infinity.' We say that $\lim f(x) = L$, as $x \to \infty$ in X, if, for any $\epsilon > 0$ there is an n such that $|f(x) - L| < \epsilon$ for all $x \in R_n$. Associated with the measures μ_n, a method of summability can be defined as follows: (μ)-$\lim f(x) = L$ as $x \to \infty$ if and only if $\lim \int f d\mu_n = L$ as $n \to \infty$.

THEOREM 2.1. *(μ)-summability is regular. If* $\lim f(x) = L$ *then* (μ)-$\lim f(x) = L$ *as* $x \to \infty$, *for* L *finite, or* $+\infty$ *or* $-\infty$.

If L is finite, we may take $L = 0$. Let $|f(x)| < \epsilon$ on R_j. Then,

$$\left| \int f\, d\mu_n \right| \leqq \int_{K_j} |f|\, d\mu_n + \int_{R_j} |f|\, d\mu_n \leqq O(1)\mu_n(K_j) + \epsilon\mu_n(R_j).$$

Hence, $\lim\sup |\int f\, d\mu_n| \leqq 0 + \epsilon$, and since ϵ is arbitrary, (μ)-$\lim f(x) = 0$ as $x \to \infty$. The remaining cases are similar.

The density of a set $S \subset X$ may be alternatively defined by means of this generalized limit operation. If S is any set having density, it is evident that $D(S) = (\mu)$-$\lim s(x)$ where $s(x)$ is the characteristic function of S. It is to be expected therefore that there should be close connections between these two. We shall say that $f(x)$ converges to L 'in density' if there is a set $A \subset X$ of zero density such that $\lim f(x) = L$, as $x \to \infty$ in $X - A$. The next few theorems will show the connection between this and summability of $f(x)$.

THEOREM 2.2. *If $f(x)$ is bounded on X, and converges to a finite limit L in density, then* (μ)-$\lim f(x) = L$.

Take $L = 0$ and suppose $|f(x)| < \epsilon$ on $R_j - A$, $|f(x)| < M$ on X. Then,

$$\left| \int f\, d\mu_n \right| \leqq \int_{K_j \cup A} |f|\, d\mu_n + \int_{R_j - A} |f|\, d\mu_n \leqq M\mu_n(K_j \cup A) + \epsilon\mu_n(R_j - A).$$

Since $D(A) = D(K_j) = 0$, we again have $\lim\sup |\int f\, d\mu_n| \leqq \epsilon$, so that (μ)-$\lim f(x) = 0$.

When L is infinite, the theorem takes a slightly different form.

THEOREM 2.3. *If $f(x)$ is bounded from below on X, and converges to $+\infty$ in density, then* (μ)-$\lim f(x) = +\infty$.

Suppose that $f(x) > C$ for $x \varepsilon R_j - A$, and $f(x) > -M$ on X. Then, as above $\int f\, d\mu_n > (-M)\mu_n(K_j \cup A) + C\mu_n(R_j - A)$ and $\liminf \int f\, d\mu_n \geqq C$. Since C may be taken arbitrarily large, $(\mu)\text{-}\lim f(x) = +\infty$.

The value of $(\mu)\text{-}\lim f(x)$ when it exists must lie between $\liminf f(x)$ and $\limsup f(x)$. Since it is in the nature of an average of the values of $f(x)$ when x is 'large,' it might be conjectured that if $(\mu)\text{-}\lim f(x)$ exists and is either of the end points of this interval, the values of $f(x)$ must themselves cluster around this value. The following theorem confirms this; the proof makes use of the additivity theorem.

THEOREM 2.4. *Let* $L = \liminf f(x)$ *be finite, and suppose that* $(\mu)\text{-}\lim f(x) = L$. *Then,* $f(x)$ *converges to* L *in density.*

We may take $L = 0$. For a given $\delta > 0$, let $A = \{x \mid f(x) > \delta\}$. Given $\epsilon > 0$, choose j so that $f(x) > -\epsilon$ on R_j. Since f is bounded on bounded sets, $|f(x)| < M$ for all x in $K(j)$. Then,

$$\int f\, d\mu_n \geqq -M\mu_n(K(j)) + \delta\mu_n(R_j \cap A) - \epsilon\mu_n(R_j),$$

as seen by splitting the integration range into $K(j)$, $R_j \cap A$, $R_j - A$. As n increases, we obtain $0 \geqq 0 + \delta\bar{D}(A) - \epsilon$ and since ϵ may be arbitrarily small, $\bar{D}(A) = D(A) = 0$. If we set $\delta = 1, 1/2, 1/3, \cdots, 1/n, \cdots$ the corresponding sets A_n form an increasing sequence of sets of zero density. Applying Theorem 1.1, let B be the modified union of the sets $\{A_n\}$, obeying the conditions: $D(B) = 0$ and $B \dot{\supset} A_n$ for all n. We show that $f(x)$ converges to 0 off B. Given ϵ, choose $n > 1/\epsilon$; since $B \dot{\supset} A_n$ there is an i with $B - K(i) \supset A_n - K(i)$. If $x \varepsilon R_i - B$ then $x \varepsilon A_n$ and $f(x) \leqq 1/n < \epsilon$. On R_j, $f(x) > -\epsilon$. If $k = \max\colon i, j\colon$, then $|f(x)| < \epsilon$ on $R_k - B$.

COROLLARY. *In this theorem, the hypothesis* $\liminf f(x) = L$ *can be relaxed to:* (i) $\inf_X f(x) > -\infty$, (ii) *for every* $\epsilon > 0$ *the set of* x *for which* $f(x) > L - \epsilon$ *has unit density.*

Again, we take $L = 0$. Let $A_\epsilon = \{x \mid f(x) > -\epsilon\}$; as ϵ decreases, these sets decrease, and by the additivity theorem, their modified intersection A is a set of unit density such that as $x \to \infty$ in A, $\liminf f(x) \geqq 0$. To prove the corollary, we apply the theorem with A replacing the whole space X. Let $f(x) > -M$ on X. Then,

$$\int_A f\, d\mu_n = \int_X f\, d\mu_n - \int_{X-A} f\, d\mu_n \leqq \int_X f\, d\mu_n + M\mu_n(X - A)$$

so that $\limsup \int_A f\,d\mu_n \leqq 0$. Since $\liminf_A f(x) \geqq 0$ we have both $\liminf_A f(x) = 0$ and $\lim \int_A f\,d\mu_n = 0$. Applying the theorem, there is a set $B \subset A$ of unit density in A and therefore of unit density in X, such that $f(x)$ converges to 0 as $x \to \infty$ in B.

In the next section, we shall give an example to show that condition (ii) by itself is not enough to imply the conclusion.

An analogous application of Theorem 1.2 can be made.

THEOREM 2.5. *Let ν be a measure on X, finite on bounded sets, and let $\int |f|\,d\nu < \infty$. Then, for any $\delta > 0$, there is a set A, $\nu(A) < \delta$, such that $f(x)$ converges to 0 as $x \to \infty$ in $X - A$.*

As before, let $A_n = \{x \mid |f(x)| > 1/n\}$. Then,

$$\nu(A_n) = n \int_{A_n} (1/n)\,d\nu \leqq n \int_{A_n} |f|\,d\nu \leqq n \int_X |f|\,d\nu < \infty.$$

The sets $\{A_n\}$ form an increasing sequence of sets of finite measure. By Theorem 1.2, there is a set A, $\nu(A) < \delta$, such that $A \dot{\supset} A_n$ for all n. From here on, the proof proceeds much as in the previous theorem.

3. Special applications. In this section, we will be concerned with a number of illustrations which arise from results of the previous section by special choices of the measures μ_n and ν involved. We take X as $\{1, 2, 3, \cdots\}$. Let ν be the measure on X which assigns to the point n the mass $1/n$. For $S \subset X$, $\nu(S)$ is finite if and only if $\Sigma\, 1/n < \infty$, where $n \in S$. Thus, sets of finite ν measure are the same as sets of "zero logarithmic density." With these substitutions, Theorem 2.5 becomes:

THEOREM 3.1. *If $\Sigma \mid c_n \mid < \infty$, then nc_n converges to 0 except for a subsequence of zero logarithmic density.* (See also [3])

This may be used in the theory of Fourier series.

COROLLARY 1. *Let $\{\phi_n\}$ be orthonormal on $[0, 1]$. Let $\lambda_n \downarrow 0$ and let $\{c_n\}$ be a sequence satisfying $\Sigma\, \lambda_n \mid c_n \mid^2 < \infty$. Set $F_n(x) = \sum_0^n c_j \phi_j(x)$. Then, for almost every x, and except for a subsequence of zero logarithmic density,*

$$F_n(x) = o(\mid n(\lambda_n - \lambda_{n+1}) \mid^{-\frac{1}{2}}).$$

In particular, if $\Sigma \mid c_n \mid^2/n^p < \infty$ for $p > 0$, $F_n(x) = o(n^{p/2})$. Starting from $\int \mid F_n \mid^2 = \sum_0^n \mid c_j \mid^2$, we have

$$\sum_0^\infty (\lambda_n - \lambda_{n+1}) \int \mid F_n \mid^2 = \sum_0^\infty (\lambda_n - \lambda_{n+1}) \sum_0^n \mid c_j \mid^2 = \sum_{j=0}^\infty \mid c_j \mid^2 \sum_{n \leqq j} (\lambda_n - \lambda_{n+1})$$

$= \Sigma \lambda_j \mid c_j \mid^2 < \infty$. Hence, for almost every x, the series $\Sigma(\lambda_n - \lambda_{n+1}) \mid F_n(x) \mid^2$ is convergent. Applying the theorem, our result follows.

COROLLARY 2. *Let $\Sigma \mid a_n \mid^2 < \infty$ and suppose that the series $\Sigma\, a_n\phi_n(x)$ is $(C, 1)$ summable to a function $g(x)$. Let $S_n(x) = \sum_0^n a_j\phi_j(x)$. Then, for almost every x and except for a subsequence of zero logarithmic density, $S_n(x)$ converges to $g(x)$* [4].

For, letting $\sigma_n(x) = (S_0 + \cdots + S_n)/(n+1)$, we have

$$(n+1)(S_n - \sigma_n) = \sum_0^n j a_j \phi_j = \Sigma\, c_j \phi_j,$$

with $\Sigma \mid c_n \mid^2/n^2 < \infty$. Applying the previous corollary in the form given above with $p = 2$, and $F_n(x) = (n+1)(S_n(x) - \sigma_n(x))$ we have $S_n - \sigma_n = o(1)$ for a. e. x and except for a subsequence of zero logarithmic density. Since $\sigma_n(x) \to g(x)$ for all x, the result follows.

Turning now to Theorem 2. 4, dealing with asymptotic density, we make the same choice of X and the measures as before, $X = \{1, 2, \cdots\}$, $K(n) = \{1, 2, \cdots, n\}$, $\mu_n(S) = \nu(S \cap K(n))/n$ where ν assigns mass 1 to each point. If $S(n)$ is the number of members of S in $K(n)$, then it is clear that $D(S) = \lim \mu_n(S) = \lim S(n)/n$, which is the usual formula for the asymptotic density of S. For convenience, we shall use "for almost all n" rather than "in density" in this instance. The corresponding notion of (μ)-summability is nothing more than $(C, 1)$ summability. Thus, Theorem 2. 4 takes the form:

THEOREM 3. 2. *Let* $\liminf S_n = L$ *be finite, and* $(C, 1)$-$\lim S_n = L$. *Then $\{S_n\}$ converges to L for almost all n.*

Similarly, the corollary to this theorem becomes:

COROLLARY 1. *If* (i) *$\{S_n\}$ is bounded from below,* (ii) *for almost all n,* $\liminf S_n \geqq L$, *and* (iii) $(C, 1) \lim S_n = L$, *then $\{S_n\}$ converges to L for almost all n.*

As stated before, condition (i) is not superfluous. Let $a_n = 1 + (-1)^n$, and let b_n be $2k - 1$ when $n = k^2$ and 0 when n is not a square. It is easily seen that $(C, 1)$-lim $a_n = (C, 1)$ lim $b_n = 1$. Setting $S_n = a_n - b_n$, we see that $(C, 1)$-lim $S_n = 0$ while $S_n \geqq 0$ for almost all n. However, both 0 and 2 are limit points of $\{S_n\}$, each of density $\frac{1}{2}$, so that $\{S_n\}$ does not converge to 0 for almost all n.

This in turn leads to a result which is quite similar to, but independent of Theorem 3. 1. This again improves on [3].

COROLLARY 2. *Let the series Σc_n converge, and suppose that nc_n is bounded from below, and that for almost all n,* $\liminf nc_n \geqq 0$. *Then, nc_n converges to zero for almost all n.*

Take

$$S_n = \sum_0^n c_j \text{ and } \sigma_n = (S_0 + S_1 + \cdots + S_n)/(n+1).$$

Since Σc_n converges, lim $(S_n - \sigma_n) = 0$ so that $(C, 1)$-lim $nc_n = 0$, and the conclusion follows.

COROLLARY 3. *Let $A = \{a_1, a_2, \cdots\}$ be a set of integers of unit density. Then, $a_{n+1} = 1 + a_n$ for almost all n.*

Set $b_n = a_{n+1} - a_n \geqq 1$. Since $\lim a_n/n = 1$, $(C, 1)$-lim $b_n = 1$ and $\{b_n\}$ converges to 1 for almost all n. However, the b_n are integral so that $b_n = 1$ for almost all n.

COROLLARY 4. *Let $f(z) = \Sigma a_n z^n$ have unit radius of convergence, and let $\Sigma \mid a_n \mid^2 < \infty$. Let* $\lim f(z) = L$ *as z approaches 1 from inside the circle $\mid z \mid < 1$. Let $S_n = \sum_0^n a_k$. Then, for almost all n, $\{S_n\}$ converges to L.*

We may take $L = 0$ by altering a_0 only. Given $\epsilon > 0$ choose δ so that if $\mid z - 1 \mid < \delta$ and $\mid z \mid < 1$, then $\mid f(z) \mid < \epsilon$. Since $f(z)/(1 - z) = \sum_0^\infty S_n z^n$, we have

$$\Sigma \mid S_n \mid^2 r^{2n} = \frac{1}{2\pi} \int_0^{2\pi} \mid f(z)/(z-1) \mid^2 d\theta \leqq \epsilon^2/(1 - r^2) + M/\delta^2$$

where $M = \sum_0^\infty \mid a_n \mid^2 \geqq \frac{1}{2\pi} \int_0^{2\pi} \mid f(z) \mid^2 d\theta$ and $\mid z \mid = r < 1$. Hence,

$$\limsup\, (1 - r^2) \Sigma \mid S_n \mid^2 r^{2n} \leqq \epsilon^2,$$

and since ϵ may be taken arbitrarily small, $\lim (1-r^2)\Sigma \mid S_n \mid^2 r^{2n}$ exists and is zero. Since Abel summability is an extension of $(C,1)$ summability, $(C,1)$-$\lim \mid S_n \mid^2 = 0$, so that by the theorem, $\{S_n\}$ converges to 0 for almost all n.

A special implication of Theorem 3. 2 is that a sequence $\{S_n\}$ which is $(C,1)$ summable to L, and for which L is either the upper or lower limit, is necessarily bounded for almost all n. It is easily seen that when L is unrestricted, $\{S_n\}$ is bounded off sets of arbitrarily small positive density. However, it is not in general possible to replace " arbitrarily small " by " zero."

THEOREM 3. 3. *There is a sequence $\{S_n\}$ of integers with $(C,1)$-$\lim S_n = 2$ which is unbounded on every set of upper density* 1.

Let A_k be the progression $(2^kN + 2^{k-1})_{N=0}^{N=\infty}$, for $k = 1, 2, \cdots$. These form a disjoint partition of the set of positive integers. Define the sequence by $S_n = k$ for all n in A_k; thus, $\{S_n\}$ begins $\langle 1, 2, 1, 3, \cdots \rangle$. As usual let $A_k(x)$ be the number of members of A_k not exceeding x. Then,

$$\sigma_m = 1/m \sum_1^m S_n = 1/m \sum_1^\infty kA_k(m) = \sum_1^{k_0} [m2^{-k} - 1/2](k/m)$$

where $k_0(m) = [\log(m/2)/\log 2] = O(\log m)$. Clearly, $\sigma_m \leqq \sum_1^\infty k2^{-k} = 2$, for every m. In the opposite direction,

$$\sigma_m \geqq \sum_1^{k_0}(m2^{-k} - 3/2)k/m \geqq \sum_1^{k_0} k2^{-k} - o(1)$$

so that as m increases, $\sigma_m \to 2$. Let M be a positive number and let B be a set of integers such that $\mid S_n \mid < M$ for all $n \,\varepsilon\, B$. B is necessarily disjoint from all the sets A_k when $k > M$. Hence,

$$\bar{D}(B) \leqq \sum_{k \leqq M} D(A_k) = \sum_{k \leqq M} 2^{-k} < 1$$

which proves the theorem.

The following simple result dealing with series of positive terms is a slight extension of a theorem of J. Arbault [1].

THEOREM 3. 4. *Let $a_n > 0$ and suppose that $\Sigma\, 1/na_n < \infty$. Let p_n be an increasing sequence of positive numbers, with $np_n = O(P_n)$ where $P_n = p_1 + p_2 + \cdots + p_n \to +\infty$. Then, $(1/P_n)\sum_1^n p_k a_k \to +\infty$.*

Since $\Sigma\, 1/na_n$ converges, $\{1/a_n\}$ converges to 0 off a set of asymptotic density zero. Let D^* be the density defined by the measure ν^* which assigns mass p_n to the point n. Any set of asymptotic density zero is also of D^* density zero, so that $\{a_n\}$ converges to $+\infty$ on a set of unit D^* density. By Theorem 2. 3, $(D^*)\text{-}\lim a_n = +\infty$ which is easily seen to be exactly the desired result. The p_n may be chosen as n^r where $r > 0$; the special choice $r = 1$ is the result obtained by Arbault.

Our next illustration is somewhat similar to the preceding. Consider the sequence-to-sequence transform defined by $\sigma_n = n^{-\rho} \sum_1^n a_k k^{\rho-1}$. When $\rho > 0$, this is convergence and boundedness preserving; however, when $\rho \leqq 0$, this may transform a convergent sequence into an unbounded sequence. We prove that if the transform is not bounded, it must be almost convergent to infinity; in particular, if the transform is bounded for almost all n, it is in fact bounded.

THEOREM 3. 5. *Let* $\sigma_n = n^{\lambda} \sum_1^n a_k/k^{\lambda+1}$ *with* $\lambda \geqq 0$, *and let* $|a_n| \leqq 1$. *Then,* σ_n *is either bounded, or* $|\sigma_n|$ *converges to* $+\infty$ *on a set* S *of upper density* 1. *Moreover, if* S *contains a sequence* $\{p_n\}$ *with* p_{n+1}/p_n *bounded, then* $\lim |\sigma_n| = +\infty$.

If $k \geqq n$, then $|\sigma_k/k^{\lambda}| \leqq |\sigma_n/n^{\lambda}| + (n+1)^{-\lambda} + \cdots + k^{-\lambda} \leqq |\sigma_n/n^{\lambda}| + (1/\lambda)(n^{-\lambda} - k^{-\lambda})$. If $R > 1$ and $n \leqq k \leqq Rn$, then $|\sigma_k| \leqq |\sigma_n| R^{\lambda} + (R^{\lambda} - 1)/\lambda$. (When $\lambda = 0$, this last term is to be replaced by $\log R$). Given $M > 0$ let $M' = MR^{\lambda} + (R^{\lambda} - 1)/\lambda$. If $|\sigma_m| > M'$, then $|\sigma_n| > M$ for all n such that $m/R \leqq n \leqq m$. Let A be the set of integers n for which $|\sigma_n| < M$; assuming that $\{\sigma_n\}$ is unbounded, choose m so that $|\sigma_m| > M'$. The interval of integers from m/R to m is free of members of A, so that $A(m) = A(m/R)$. Dividing by m/R and letting m increase along the sequence for which σ_n is unbounded, we obtain $R\underline{D}(A) \leqq \bar{D}(A) \leqq 1$. Letting R increase, we have $\underline{D}(A) = 0$. As M increases, so do the sets A; appealing to the additivity theorem, the modified union of the sets A is a set B such that $\underline{D}(B) = 0$ while for any M, $|\sigma_n| > M$ for all sufficiently large integers n not in B. Thus, $|\sigma_n|$ converges to $+\infty$ on the complement S of B. S has upper density 1. If there is a sequence $\{p_n\}$ such that $p_{n+1}/p_n = O(1)$ and $|\sigma_{p_n}| \to +\infty$ then by the argument above, the intervals $[p_n/R, p_n]$ are disjoint from the set A for all sufficiently large n. If R is chosen so that $R > p_{n+1}/p_n$ for all n, then these intervals overlap, and the set A is finite. Since this is true for any choice of M, $|\sigma_n|$ converges to $+\infty$.

4. Complex values. Most of the theorems of Section **2** concerning the behavior of $f(x)$ as $x \to \infty$ in X go over to complex-valued functions; one is of sufficient interest to require separate treatment, namely Theorem 2.4, which asserts that a function which is summable to its limit superior or inferior must converge to that limit point on a set of unit density. We shall prove the complex form of this theorem, restricting ourselves for simplicity to the case of sequences and $(C, 1)$ summability. It is first necessary to choose a correct replacement for upper and lower limits for a complex sequence $\{S_n\}$. We find this in the notion of the core of a sequence, as introduced by Knopp. Let C_n be the closed convex hull of the infinite set $\{S_n, S_{n+1}, S_{n+2}, \cdots\}$ and let $C = \bigcap C_n$.

This closed convex set is called the core of $\{S_n\}$. It contains the convex hull of the set of limit points of $\{S_n\}$ and coincides with this set, if $\{S_n\}$ is bounded. In many theorems about summability of complex sequences, the core replaces the oscillation set of a real sequence. For example, if $\sigma_n = (S_1 + \cdots + S_n)/n$ the core of $\{\sigma_n\}$ is a subset of the core of $\{S_n\}$. We recall that a boundary point of a convex set is called extreme if it is not the mid point of two other points of the set. We introduce the term *outer limit point* for any of the extreme points of the core of a sequence $\{S_n\}$. It is easily seen that these are in fact limit points of $\{S_n\}$. When $\{S_n\}$ is real, the outer limit points are merely the upper and lower limits. For a bounded sequence, the core may then be described as the convex hull of its set of outer limit points.

THEOREM 4.1. *Let $\{S_n\}$ be a bounded complex sequence, and let it be Cesaro summable to one of its outer limit points, p. Then, $\{S_n\}$ converges to p for almost all n.*

We may assume that $p = 0$ and that the line through p which supports the core C is the real axis, and lies below C. Let $S_n = x_n + iy_n$; for any $\delta > 0$, all but a finite number of these lie within δ of C. We have $(C, 1)$-$\lim x_n = (C, 1)$-$\lim y_n = 0$ and $\liminf y_n \geqq 0$. By Theorem 3.2, $\{y_n\}$ converges to 0 for almost all n. Hence, for any $\epsilon > 0$, the set of n for which $y_n > \epsilon$ has zero density. Since 0 is an extreme point of C, the set C can touch the real axis on one side of the origin only. We suppose that the left side of C lies above this axis. Then, x_n is bounded from below, and for any $\delta > 0$, $x_n > -\delta$ for almost all n. Appealing to the stronger form, Corollary 1 of Theorem 3.2, $\{x_n\}$ converges to 0 for almost all n. Since the intersection of two sets of unit density has unit density, $\{S_n\}$ converges to 0 for almost all n.

We do not know if this theorem holds for unbounded sequences $\{S_n\}$. A slight modification of the above proof shows that the Theorem is valid in any case, if p is a *regular* extreme point of C, i. e. one for which there is a supporting line at p which contacts C nowhere else. In this case the Theorem also holds for a sequence $\{S_n\}$ of points in a Banach space, if in addition it is assumed that the closure of the set $\{S_1, S_2, \cdots\}$ is compact. The argument via real and imaginary parts of S_n may be replaced by the use of the functional F which supports C at p with $F(p) = 1$; one can immediately infer that $F(S_n)$ converges to 1 for almost all n. If a neighborhood of p is deleted, it is easily seen that the remaining S_n form a subsequence of zero density. This argument does not seem suited to the more precise theorem above. In the case of a general density, and a complex valued function $f(x)$, the core of f is to be taken as the intersection of the closed convex hulls of the sets $f(R_n)$.

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SQUARE SUMMATION AND LOCALIZATION OF DOUBLE TRIGONOMETRIC SERIES.*

By VICTOR L. SHAPIRO.

1. Introduction. Let $\sum a_M e^{iMX}$ be a double trigonometric series where a_M are arbitrary complex numbers and where $M = (m, n)$, $X = (x, y)$, $MX = mx + ny$, and $|M| = \max(|m|, |n|)$. The series will be said to be square convergent at a point X if the square partial sums of rank R

$$(1) \qquad S_R(X) = \sum_{|M| \leqq R} a_M e^{iMX}$$

converge to the finite value $L(X)$. The series will be said to be square summable (C, ρ), $\rho > 0$, to the sum $L(X)$ if the (C, ρ) square means of rank R,

$$(2) \qquad \sigma_R^{(\rho)}(X) = 2\rho R^{-2\rho} \int_0^R S_r(X)(R^2 - r^2)^{\rho-1} r\,dr$$

$$= \sum_{|M| \leqq R} a_M e^{iMX}(1 - |M|^2/R^2)^\rho = \sum_{r=0}^{[R]} (1 - r^2/R^2)^\rho \sum_{|M|=r} a_M,$$

converge to the finite value $L(X)$.

It is the purpose of this paper to study the localization theory of double trigonometric series for square summation. We shall use for this study the process of formal multiplication of series developed by Rajchman and Zygmund [2]. A comparison of the results obtained in this paper with those obtained by Berkovitz [1] for circular summation shows a decided difference between the two methods.

2. Definitions and notation. The notation in this paper will be for the most part vectorial, thus for example the index pair (m, n) will be designated by M, the capital letter of the first letter occurring, and $M + X$ will stand for $(m + x, n + y)$.

By $a_M = o[(|m| + 1)^\gamma(|n| + 1)^\eta]$ will be meant the following: Given an $\epsilon > 0$, there exists an $R(\epsilon)$ such that if $|M| > R(\epsilon)$, then

$$|a_M| < \epsilon(|m| + 1)^\gamma(|n| + 1)^\eta.$$

$a_M = O[(|m| + 1)^\gamma(|n| + 1)^\eta]$ will be defined in a similar manner.

* Received November 19, 1952; revised February 12, 1953.

Letting $T = \sum a_M e^{iMX}$, we set $\delta^j T/\delta x^j = \sum (im)^j a_M e^{iMX}$ ($\delta^0 T/\delta x^0$ will be interpreted as T); so that $\delta/\delta x$ is the symbol of partial differentiation ($= \partial/\partial x$).

We shall designate the fundamental square $[(x, y); 0 \leqq x \leqq 2\pi, 0 \leqq y \leqq 2\pi]$ by Ω.

$f(X)$ will be said to be of class $C^{(k)}$ if all its partial derivatives up to and including those of order k exist and are continuous.

3. Formal multiplication. Let $T_1 = \sum a_M e^{iMX}$, $T_2 = \sum \alpha_M{}^{iMX}$ be two double trigonometric series. We define their formal product $T_1 T_2 = T_3$ to be the series $T_3 = \sum A_M e^{iMX}$, where $A_M = \sum_P a_P \alpha_{M-P}$. The definition only makes sense when the A_M, called the formal product coefficients, are defined. In particular, if the a_M are bounded and $\sum_M |\alpha_M| < \infty$, then A_M is defined for every M. We prove the following lemma regarding formal product coefficients.

LEMMA 1. *Suppose that T_1 is a series with coefficients $a_M = o(|M|^\rho)$, $\rho \geqq 0$, and T_2 is a series with coefficients such that $\sum |\alpha_M|\,|M|^\rho < \infty$. Then A_M is defined for all M and A_M is $o(|M|^\rho)$.*

For $|A_M| \leqq (\sum_{|P|<|M|/2} + \sum_{|M|/2 \leqq |P| \leqq 2|M|} + \sum_{|P|>2|M|}) |a_P \alpha_{M-P}|$. It is easy to see that the first and third sum are $o(1)$ and the second sum is $o(|M|^\rho)$, which proves the lemma.

In proving the basic theorems concerning formal products, certain lemmas will be required. We shall prove them first.

LEMMA 2. *Let $\sum \alpha_M e^{iMX} = 0$ for X in a set E where*

$$\alpha_M = O[(|m| + 1)^{-\theta}(|n| + 1)^{-\theta}], \qquad \theta > 1.$$

Then there is a $K > 0$ such that

(i) $\sum_{|M| \leqq R} |\alpha_M| \leqq K(R + 1)^{1-\theta}$;

(ii) $|\sum_{|M| \geqq R} \alpha_{M-P} e^{i(M-P)X}| \leqq K(|R - |P|| + 1)^{1-\theta}$ *uniformly for X in E;*

(iii) *if $p \geqq q \geqq 0$, then* $\sum_{|M| \leqq R,\, n^2 > m^2} |\alpha_{M-P}| \leqq K(p - q + 1)^{1-\theta}$;

(iv) *if $p \geqq q \geqq 0$, then, uniformly for X in E,*

$$|\sum_{|M| \leqq R,\, m^2 \geqq n^2} \alpha_{M-P} e^{i(M-P)X}| \leqq K(p - q + 1)^{1-\theta} + K(|R - p| + 1)^{1-\theta}.$$

To prove (i), we notice that

$$(3) \qquad \sum_{|M| \geqq R} |\alpha_M| \leqq \sum_{i=0} \sum_{|M|=[R]+i} |\alpha_M|.$$

Observing also that the inner sum on the right of (3) is $O[(R+i)^{-\theta}]$, we have that the right side of (3) is $O(R^{1-\theta})$.

To prove (ii), we can assume, since $\sum |\alpha_M|$ is convergent, that

$$|P| \geqq R+1 \quad \text{or} \quad |P| \leqq R-1.$$

If the former holds then

$$\left| \sum_{|M| \leqq R} \alpha_{M-P} e^{i(M-P)X} \right| \leqq \sum_{|M| \geqq |P|-R} |\alpha_M|, \text{ while}$$

$$\left| \sum_{|M| \leqq R} \alpha_{M-P} e^{i(M-P)X} \right| \leqq \sum_{|M| > R-|P|} |\alpha_M|$$

if the latter holds, because the set $[M - P; |M| > R]$ lies outside of a square with center at the origin and side of length $R - |P|$. Applying (i) of this lemma to both cases completes the proof of (ii).

To prove (iii), we may suppose that $p > q + 2$. Observing that the set $[M - P; m^2 < n^2 \leqq R^2]$ lies outside of a square with center at the origin and side of length $(p - q)/2$, we have that

$$\sum_{|M| \leqq R,\, n^2 > m^2} |\alpha_{M-P}| \leqq \sum_{|M| \geqq |p-q|/2} |\alpha_M|.$$

Applying (i) of this lemma, we obtain the desired result.

(iv) is an immediate consequence of (ii) and (iii).

Lemma 3. *If* $a_M = o[(|m|+1)^{-\gamma}(|n|+1)^{-\eta}]$ *where* $0 \leqq \gamma + \eta \leqq 1$, $0 \leqq \gamma < 1$, *and* $0 \leqq \eta < 1$, *then* $\sum_{|M|=i} |a_M| = o[i^{1-(\gamma+\eta)}]$.

This lemma follows in an obvious manner from the fact that

$$\sum_{n=-i}^{i} |a_{in}| = o(i^{-\gamma} \sum_{n=-i}^{i} (|n|+1)^{-\eta}) = o[i^{1-(\gamma+\eta)}].$$

We are now in a position to prove the following two theorems concerning formal products.

Theorem 1. *Let T_1 and T_2 be two double trigonometric series where $T_1 = \sum a_M e^{iMX}$ and $T_2 = \sum \alpha_M e^{iMX}$ with the following properties:*

(i) $a_M = o[(|m|+1)^{-\gamma}(|n|+1)^{-\eta}]$, $\gamma + \eta = 1$; $\gamma, \delta\eta \geqq 0$;

(ii) $\alpha_M = O[(|m|+1)^{-3}(|n|+1)^{-3}]$;

(iii) $\sum \alpha_M e^{iMX} = 0$ *for X belonging to a plane set E.*

Then the formal product T_3 of T_1 and T_2 is uniformly square convergent to zero for X in E.

THEOREM 2. *Let T_1 and T_2 be two double trigonometric series where $T_1 = \sum a_M e^{iMX}$ and $T_2 = \sum \alpha_M e^{iMX}$ with the following properties:*

(i) $a_M = o[(|m| + 1)^{-\gamma}(|n| + 1)^{-\eta}]$, $0 \leqq \gamma + \eta < 1$; $\delta\eta \geqq 0$;

(ii) $\alpha_M = O[(|m| + 1)^{-5}(|n| + 1)^{-5}]$;

(iii) $\delta^j T_2/\delta x^j = 0$, $\delta^k T_2/\delta y^k = 0$, $j = 0, 1, 2$, $k = 0, 1, 2$, *for X belonging to a plane set E. Then the formal product T_3 of T_1 and T_2 is uniformly square summable $(C, 1 - (\gamma + \eta))$ to zero for X in E.*

Remark 1. It will be apparent from the proofs of the theorems that (ii) in Theorem 1 can be replaced by $\alpha_M = O[(|m| + 1)^{-(2+\epsilon)}(|n| + 1)^{-(2+\epsilon)}]$ and that (ii) in Theorem 2 can be replaced by

$$\alpha_M = O[(|m| + 1)^{-(4+\epsilon)}(|n| + 1)^{-(4+\epsilon)}], \text{ where } \epsilon > 0.$$

That the formal product coefficients are defined in both theorems follows from Lemma 1. For simplicity of notation in the proofs of both theorems we shall suppose that 0 is in E and give the proof of the theorem only for this point. From the method of proof the uniformity of convergence or summability for all points in E will follow automatically.

Set

$$(4) \quad \sum_{|M| \leqq R} A_M(1 - |M|^2/R^2)^\beta$$
$$= \Big(\sum_{|P|=0}^{[R]-1} \sum_{|P|=[R]}^{[R]+1} \sum_{|P|=[R]+2}^{[2R]} \sum_{|P|=[2R]+1}^{\infty}\Big) a_P \sum_{0 \leqq |M| \leqq R} \alpha_{M-P}(1 - |M|^2/R^2)^\beta$$
$$= A + B + C + D, \text{ where } \beta = 0 \text{ or } 1.$$

Then we see from (i) of Lemma 2 that under the conditions of either Theorem 1 or 2,

$$(5) \qquad |D| \leqq \sum_{|P|>2R} |a_P| \sum_{|M| \leqq R} |\alpha_{M-P}| = o(R^{-1}).$$

Setting $\beta = 0$ in (4) and bearing (5) in mind, we observe that to prove Theorem 1 it only remains to show that A, B, and C are $o(1)$. But these facts follow easily from (i) and (ii) of Lemma 2 and Lemma 3, for by these lemmas, $|A|$, $|B|$, $|C|$ are majorized by

$$\sum_{i=0}^{[R]-1} O[(R-i)^{-2}] \sum_{|P|=i} |a_P|, \quad \sum_{i=[R]}^{[R]+1} O(1) \sum_{|P|=i} |a_P|, \quad \sum_{i=[R]+2}^{[2R]} O[(i-R)^{-2}] \sum_{|P|=i} |a_P|,$$

respectively, and each of these sums is $o(1)$.

To prove Theorem 2, we shall first show that under the conditions of this theorem the left side of (4) is $o(R^{-(\gamma+\eta)})$ when $\beta = 1$. To do this, by a simple argument of symmetry, it is sufficient to consider a_P equal to zero except for points in the first octant, i. e., $a_P = 0$ unless $p \geqq q \geqq 0$.

Applying the equality $m^2 = p^2 + 2p(m-p) + (m-p)^2$, we observe that with $\beta = 1$ the inner sum on the right side of (4) can be written as

$$\begin{aligned} R^{-2} \sum_{|M|\leqq R,\, n^2 \leqq m^2} \alpha_{M-P}[R^2 - p^2 - 2p(m-p) - (m-p)^2]^\beta \\ + R^{-2} \sum_{|M|\leqq R,\, n^2 > m^2} \alpha_{M-P}[R^2 - p^2 + p^2 - q^2 - 2q(n-q) - (n-q)^2]^\beta, \end{aligned} \tag{6}$$

and consequently, from Lemmas 2 and 3, that $|A|$, $|B|$, $|C|$ are majorized by

$$R^{-1} \sum_{p=0}^{[R]-1} o(p^{1-(\gamma+\eta)})(R-p)^{-2} + R^{-2} \sum_{p=0}^{[R]-1} o(p^{1-(\gamma+\eta)}),$$

$$R^{-1} \sum_{p=[R]}^{[R]+1} p^{1-(\gamma+\eta)}, \qquad R^{-2} \sum_{p=[R]+2}^{[2R]} o(p^{2-(\gamma+\eta)})(p-R)^{-3} + o(p^{1-(\gamma+\eta)}),$$

respectively; hence each of them is $o(R^{-(\gamma+\eta)})$.

From these last three facts and (5), we conclude that with $\beta = 1$ the left side of (4) is $o(R^{-(\gamma+\eta)})$ and consequently that

$$\sigma_R^{(1)}(0) = o(R^{-(\gamma+\eta)}). \tag{7}$$

Theorem 2 is thus proved in the special case when $\gamma + \eta = 0$. Let us suppose for the rest of the proof that $0 < \gamma + \eta < 1$.

Observing that for an integer j,

$$S_j(0) = (2j+1)^{-1}[(j+1)^2\sigma_{j+1}^{(1)}(0) - j^2\sigma_j^{(1)}(0)]$$

we conclude from (7) that

$$S_R(0) = o(R^{1-(\gamma+\eta)}). \tag{8}$$

By (2) and (8), in order to complete the proof of Theorem 2, it remains only to show that

$$2(1-\gamma-\eta)R^{-2(1-\gamma-\eta)} \int_0^{R-1} S_r(0)(R^2 - r^2)^{-(\gamma+\eta)} r\,dr = o(1). \tag{9}$$

Observing, however, that by (7), $\int_0^R S_r(0)r\,dr = o(R^{2-(\gamma+\eta)})$, we conclude after integrating the integral in (9) by parts that this integral is $o(R^{2-2(\gamma+\eta)})$ and consequently that (9) holds, which gives us Theorem 2.

Remark 2. From the proofs of Theorems 1 and 2 it is evident that if the α_M were functions of X such that in Theorem 1,

$$\alpha_M(X) = O[(|m|+1)^{-3}(|n|+1)^{-3}]$$

uniformly in X and in Theorem 2,

$$\alpha_M(X) = O[(|m|+1)^{-5}(|n|+1)^{-5}]$$

uniformly in X, both theorems would still hold.

Unlike circular summation (see Berkovitz [1], p. 330), the formal product theorems using square summation cannot be extended to higher orders of summability. We show this with the following theorem:

THEOREM 3. *Given any* $\epsilon > 0$ *and any integer* $k > 0$ *there exists two double trigonometric series* $T_1 = \sum a_M e^{iMX}$ *and* $T_2 = \sum \alpha_M e^{iMX}$ *with the following properties:*

(i) $a_M = o(|M|^\epsilon)$;

(ii) $\alpha_M = 0$ *except for a finite number of* M;

(iii)) $\delta^{i+j}T_2/\delta x^i \delta y^j = 0,\ 0 \leqq i+j \leqq k$ *for* $X = 0$,

such that the formal product T_3 *of* T_1 *and* T_2 *is square summable* (C,ρ), $\rho \geqq 0$, *to infinity for* $X = 0$.[1]

We choose for our series $T_1 = \sum_{m=2}^{\infty} b_m e^{imx}(e^{imy} + e^{i(m+1)y})/2$ where $b_j = j^\epsilon (\log j)^{-1}$, $j = 2, 3, 4, \cdots$ and $T_2 = (1 - e^{ix})^{k+1}$. Then

$$a_{mn} = b_m/2 \text{ if } n = m \text{ or } n = m+1 \text{ and } m \geqq 2; \text{ otherwise } a_{mn} = 0,$$

$$\alpha_{mn} = (-1)^m \binom{k+1}{m},\ 0 \leqq m \leqq k+1 \text{ and } n = 0; \text{ otherwise } \alpha_{mn} = 0.$$

Clearly conditions (i), (ii), and (iii) of the theorem are satisfied.

Now

$$A_M = \sum_P a_P \alpha_{M-P} = (b_{n-1}\alpha_{m-(n-1)\,0} + b_n \alpha_{m-n\,0})/2 \text{ if } n \geqq 2; \text{ otherwise } A_M = 0.$$

By (2), the theorem will be proved if it is shown that

$$S_R(0) = \sum_{|M| \leqq R} A_M \to \infty.$$

A short calculation shows that, for t an integer,

$$S_{t+k+1}(0) = \sum_{j=1}^{k} b_{t+k} \sum_{m=0}^{k+1-j} \alpha_{m\,0} + b_{t+k+1}/2$$

$$= (t+k+1)^\epsilon [\log(t+k+1)]^{-1} K + o(t^{\epsilon-1}), \text{ where } K = \tfrac{1}{2} + \sum_{j=1}^{k} \sum_{m=0}^{k+1-j} \alpha_{m\,0}.$$

[1] The author is indebted to the referee for suggesting a short proof to this theorem.

It is clear that $K \neq 0$, and consequently, $S_{t+k+1}(0) \rightarrow +\infty$ or $S_{t+k+1}(0) \rightarrow -\infty$ according to the sign of K.

We end this section on formal products with the following theorems:

THEOREM 4. *Let T_1 and T_2 be two double trigonometric series satisfying the conditions of Theorem 1, except that T_2 converges to a function $\lambda(X)$ which need not be zero. Then the series*

$$T_3 - \lambda T_1 = \sum A_M e^{iMX} - \lambda(X) \sum a_M e^{iMX}$$

is uniformly square convergent to zero for all X.

THEOREM 5. *Let T_1 and T_2 be two double trigonometric series where $T_1 = \sum a_M e^{iMX}$ and $T_2 = \sum \alpha_M e^{iMX}$ with the following properties:*

(i) $a_M = o[(|m|+1)^{-\gamma}(|n|+1)^{-\eta}]$, $0 \leqq \gamma + \eta < 1$; $\gamma, \eta, \geqq 0$;

(ii) $\alpha_M = O[(|m|+1)^{-5}(|n|+1)^{-5}]$;

(iii) $\delta^j T_2/\delta x^j = 0$, $\delta^k T_2/\delta^k = 0$, $j = 1, 2$, $k = 1, 2$ *for X in a set $E \subset \Omega$, the fundamental square. Designate the formal product of T_1 and T_2 by T_3 and the function to which T_2 converges on E by $\lambda(X)$. Then the series $T_3 - \lambda T_1$ is uniformly square summable $(C, 1 - (\gamma + \eta))$ to zero for X in E.*

We shall only give the proof for Theorem 4, Theorem 5 being proved in the same manner from Theorem 2.

Define $T_2{}^*$ to be the trigonometric series $\sum \alpha_M{}^* e^{iMX}$ where $\alpha_M{}^* = \alpha_M$ if $|M| \neq 0$, $\alpha_0{}^* = \alpha_0 - \lambda(X)$. Since $\lambda(X)$ is a bounded function, $\alpha_M{}^*(X) = O[(|m|+1)^{-3}(|n|+1)^{-3}]$ uniformly for all X. Setting $A_M{}^* = \sum a_P \alpha_{M-P}{}^* = A_M - \lambda(X) a_M$, we have by Theorem 1 and Remark 2 that $\sum_{|M| \leqq R} A_M{}^* e^{iMX}$ converges to zero uniformly for all X.

4. Localization. We shall now apply the results of the formal product theorems to the problem of localization. This application will be prefaced, however, by a few remarks.

Remark 3. If $f(X)$ is in L and periodic of period 2π in each variable, we may associate to f its Fourier series. Thus $f \sim \sum c_M e^{iMX} = \mathfrak{S}[f]$. In what follows $\mathfrak{S}[f]$ will be used to denote the Fourier series of a function f. The square partial sums $S_R(X)$ of $\mathfrak{S}[f]$ are given by

$$S_R(X) = \sum_{|M| \leqq R} c_M e^{iMX} = \pi^{-2} \int_0^{2\pi} \int_0^{2\pi} f(U) D_R(u-x) D_R(v-y)\, du\, dv,$$

where $D_R(t) = 1/2 \sum_{m=-[R]}^{[R]} e^{imt}$ is the Dirichlet kernel.

Remark 4. Let T be the double trigonometric series, $T = \sum c_M e^{iMX}$. An operator L^2, which will be called the double integral operator, is defined to act on T as follows:

$$L^2T = \frac{c_0 x^2 y^2}{4} + \frac{y^2}{2} \sum_{m=-\infty}^{\infty}{}' c_{m0}(im)^{-2} e^{imx}$$
$$+ \frac{x^2}{2} \sum_{n=-\infty}^{\infty}{}' c_{0n}(in)^{-2} e^{iny} + \sum_M{}'' c_M(mn)^{-2} e^{iMX}$$

where $'$ indicates the omission of the value zero, and $''$ indicates the omission of the values $(0, n)$ and $(m, 0)$. It is clear that L^2T converges uniformly to a function $F(X)$ if $c_M = o(1)$. We shall call $F(X)$ the function associated with T.

Remark 5. We shall now discuss another essential notion, that of a localizing function. Let $\mathfrak{R}$ be a closed domain contained in the interior of the fundamental square Ω. Let $\mathfrak{R}^0$ denote the interior of $\mathfrak{R}$, and $\mathfrak{R}'$ be another closed domain such that $\mathfrak{R}' \subset \mathfrak{R}^0$. A function $\lambda(X)$ which is continuous, of period 2π in each variable, has Fourier coefficients $O(|M|^{-\mu})$, μ being a sufficiently large positive integer, and such that $\lambda(X) = 0$ for X not in $\mathfrak{R} (\text{mod } 2\pi)$ and $\lambda(X) = 1$ for X in $\mathfrak{R}'(\text{mod } 2\pi)$, is called a localizing function for the domains $\mathfrak{R}$ and $\mathfrak{R}'$. That such a function can be constructed for given closed domains $\mathfrak{R}$ and $\mathfrak{R}'$ is a well known fact.

We now state the theorem from which we deduce our localization theorem.

THEOREM 6. *Let T_1 be a double trigonometric series with coefficients $a_M = o[(|m|+1)^{-\gamma}(|n|+1)^{-\eta}]$, $0 \leqq \gamma + \eta \leqq 1$, $0 \leqq \gamma < 1$, $0 \leqq \eta < 1$. Then the series L^2T_1 converges uniformly to a function $F(X)$. Furthermore let $\lambda(X)$ be a localizing function of class $C^{(14)}$ associated with the domains $\mathfrak{R}$ and $\mathfrak{R}'$ and whose Fourier coefficients are consequently*

$$\alpha_M = O[(|m|+1)^{-7}(|n|+1)^{-7}].$$

Then the difference

$$(10) \quad \Delta_R(X) = \sum_{|M| \leqq R} a_M e^{iMX}$$
$$- \pi^{-2} \int_0^{2\pi} \int_0^{2\pi} F(U)\lambda(U) d^2D_R(u-x)/dx^2 d^2D_R(v-y)/dy^2 \, du dv$$

is uniformly summable $(C, 1-(\gamma+\eta))$ to zero for X in $\mathfrak{R}'$.

By Remark 4,

$$F(X) = a_0x^2y^2/4 + y^2/2 \sum_{m=-\infty}^{\infty}{}' \, a_{m0}(im)^{-2}e^{imx}$$

$$+ x^2/2 \sum_{n=-\infty}^{\infty}{}' \, a_{0n}(in)^{-2}e^{iny} + \sum_{M}{}'' \, a_M(mn)^{-2}e^{iMX}$$

$= F_4 + F_3 + F_2 + F_1$. We shall consider four different cases in the proof corresponding to F_1, F_2, F_3, and F_4.

For case 1, we assume that $a_M = 0$ if $m = 0$ or $n = 0$. Then F, in this case, is identical with F_1 and the right side of (10) is the square partial sum of rank R of the series $T_1 - \delta^4\mathfrak{S}[F\lambda]/\delta x^2\delta y^2$. By Theorem 4, the formal product $\mathfrak{S}[F]\mathfrak{S}[\lambda] = \mathfrak{S}[F\lambda]$. Let C_M designate the Fourier coefficients of this formal product and let d_M designate the coefficients of $\mathfrak{S}[\lambda]$. Then

$$m^2n^2C_M = \sum_{P}{}'' \, a_P d_{M-P}(pq)^{-2}[(m-p)^2 + 2p(m-p) + p^2]$$
$$\times \, [(n-q)^2 + 2q(n-q) + q^2]$$

and consequently $\Delta_R(X)$ is equal to the square partial sum of rank R of the series

$$(11) \quad T_1 - T_1\mathfrak{S}[\lambda] - \sum_{\substack{k=0 \\ j+k\neq 4}}^{2} \sum_{j=0}^{2} \rho_{jk}\delta^{j+k}\mathfrak{S}[F]/\delta x^j\delta y^k\delta^{4-(j+k)}\mathfrak{S}[\lambda]/\delta x^{2-j}\delta y^{2-k}$$

where ρ_{jk} are constants. Now by Theorem 4 or Theorem 5 the partial sum of rank R of the series $T_1 - T_1\mathfrak{S}[\lambda]$ and of each of the series in the sum in (11) is uniformly summable $(C, 1-(\gamma+\eta))$ to zero for X in $\mathfrak{R}'$, and the proof of case 1 is complete.

For case 2, we assume that $a_0 = 0$ and $a_M = 0$ if $n \neq 0$. Then

$$T_1 = \sum_{m=-\infty}^{\infty}{}' \, a_{m0}e^{imx} \quad \text{and} \quad F(X) = G(x)y^2/2 = F_2(X),$$

where

$$G(x) = \sum_{m=-\infty}^{\infty}{}' \, a_{m0}(im)^{-2}e^{imx},$$

and furthermore, we observe that the right side of (10) is given by $T_1 - \delta^4\mathfrak{S}[G\lambda y^2/2]/\delta x^2\delta y^2$ and that the formal product $\mathfrak{S}[G]\mathfrak{S}[\lambda y^2/2] = \mathfrak{S}[G\lambda y^2/2]$. Proceeding as in case 1, we obtain that $\Delta_R(X)$ is the square partial sum of rank R of the series

$$(12) \quad T_1 - T_1\mathfrak{S}[\delta^2(\lambda y^2/2)/\delta y^2]$$
$$- \sum_{i=0}^{i} \rho_i\delta^i\mathfrak{S}[G]/\delta x^i\delta^{2-i}\delta^2\mathfrak{S}[\lambda y^2/2]/\delta x^{2-i}\delta y^2$$

where ρ_i are constants. Noticing that $\delta^2(\lambda y^2/2)/\delta y^2 = 1$ on $\mathfrak{R}'$, we conclude from Theorem 4 or Theorem 5 that the square partial sum of rank R of the series $T_1 - T_1\mathfrak{S}[\delta^2(\lambda y^2/2)/\delta y^2]$ and of each of the series in the sum in (12) is uniformly $(C, 1-(\gamma+\eta))$ summable to zero for X in $\mathfrak{R}'$. This completes the proof for case 2.

For case 3, we assume that $a_0 = 0$ and $a_M = 0$ if $n \neq 0$. Then $T_1(X) = \sum_{n=-\infty}^{+\infty} a_{0n}e^{iny}$ and $F(X) = F_3(X)$, and by an argument similar to case 2, we conclude that $\Delta_R(X)$ is uniformly summable $(C, 1-(\gamma+\eta))$ to zero for X in $\mathfrak{R}'$.

For case 4, we assume that $a_M = 0$ when $M \neq 0$. Then $T_1 = a_0$ and $F(X) = a_0x^2y^2/4$, and consequently the right side of (10) is the square partial sum of rank R of the series $T_1 - \delta^4\mathfrak{S}[\lambda a_0x^2y^2/4]/\delta x^2\delta y^2$, which clearly converges to zero uniformly for X in $\mathfrak{R}'$.

Putting cases 1, 2, 3, and 4 together, we have the proof of the theorem, for T_1 can be considered a sum of four parts, one corresponding to each case.

It is at this point that the divergence in localization between square summation and circular summation can be seen. Theorem 6 cannot be extended to trigonometric series whose coefficients are $o(|M|^\epsilon)$, $\epsilon > 0$, by means of formal products because the key theorem in the proof is Theoerm 5, and Theorem 3 gives a direct contradiction to Theorem 5 for such coefficients. On the other hand, as Berkovitz [1] shows, no such difficulty exists in circular summation where the formal product theorems exist regardless of the order of the coefficients, and consequently, for circular summation, localization goes through.

Using Theorem 6, we can now state the main theorem for localization in square summation.

THEOREM 7. *If T and T' are two double trigonometric series with coefficients $o[(|m|+1)^{-\gamma}(|n|+1)^{-\eta}]$, $0 \leqq \gamma+\eta \leqq 1$, $0 \leqq \gamma < 1$, $0 \leqq \eta < 1$ and if the functions F and F' associated with T and T' are equal in a closed domain $\mathfrak{R}$ contained in the interior of the fundamental square Ω, then in every smaller closed domain $\mathfrak{R}'$ contained in $\mathfrak{R}^0$, the interior of $\mathfrak{R}$, the series $T - T'$ is uniformly square summable $(C, 1-(\gamma+\eta))$ to zero. The condition that $F - F'$ vanish in $\mathfrak{R}$ can be replaced by the condition that*

$$\int_0^{2\pi}\int_0^{2\pi} \lambda(U)[F(U) - F'(U)]d^2D_R(u-x)/dx^2\, d^2D_R(v-y)/dy^2\, dudv$$

be uniformly summable $(C, 1-(\gamma+\eta))$ to zero in every $\mathfrak{R}'$ where $\lambda(X)$ is a localizing function for $\mathfrak{R}$ and $\mathfrak{R}'$ of class $C^{(14)}$. In particular, this latter

result is alway true if $F - F'$ is a function of class $C^{(8)}$ in a domain $\mathfrak{R}_1^0$ containing $\mathfrak{R}$ and $\delta^4(F - F')/\delta x^2 \delta y^2 = 0$ for X in $\mathfrak{R}^0$.

The last statement of the theorem follows from the fact that

$$\pi^{-2} \int_0^{2\pi} \int_0^{2\pi} \lambda(U)[F(U) - F'(U)] d^2 D_R(u - x)/dx^2 \, d^2 D_R(v - y)/dy^2 \, du dv$$

is the square partial sum of rank R of

$$\delta^4 \mathfrak{S}[(F - F')\lambda]/\delta x^2 \delta y^2 = \mathfrak{S}[\delta^4 (F - F')\lambda/\delta x^2 \delta y^2].$$

This latter Fourier series has coefficients $o[(|m| + 1)^{-2}(|n| + 1)^{-2}]$, and has square partial sums which converge to zero uniformly for X in $\mathfrak{R}'$.

The rest of the theorem follows immediately from Theorem 6.

It is to be noticed, in closing, that for the orders of Cesaro summability discussed, localization by squares except for the $(C, 1)$ case requires a weaker hypothesis then localization by circles since the condition $o[(|m| + 1)^{-\gamma}(|n| + 1)^{-\eta}]$ is weaker than $o[(m^2 + n^2)^{-(\gamma+\eta)/2}]$ for the values of γ and η considered in this paper.[2]

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[2] The present paper is based on Part I of the author's doctoral dissertation submitted to the Mathematics Department of the University of Chicago. The author wishes to acknowledge his indebtedness to Professor Antoni Zygmund, under whose direction the dissertation was written.

AN IDEAL-THEORETIC CHARACTERIZATION OF THE RING OF ALL LINEAR TRANSFORMATIONS.*

By KENNETH G. WOLFSON.

Introduction. It is well known that a simple ring that satisfies the minimum condition on right (left) ideals is isomorphic to the complete ring of linear transformations of a finite-dimensional vector space ([8], p. 67). This determination of the structure of such simple rings also serves as an abstract characterization of the ring of all linear transformations of a finite-dimensional space. In studying the structure theory of rings not restricted by the minimum condition, emphasis has been placed on the notion of a primitive ring ([10]), a ring which is isomorphic to a dense ring of linear transformations of an (in general) infinite-dimensional vector space. Although necessary and sufficient conditions have been given ([9], [10]) that an abstract ring be isomorphic to a dense ring, no characterization of the complete ring of linear transformations of an infinite-dimensional vector space is to be found in the literature. This ring of linear transformations also arises in projective geometry since the principal left ideals of this ring form a lattice which is projectively equivalent to the projective geometry determined by the lattice of subspaces of the underlying vector space ([4], p. 173 and [17], p. 6). In fact, if the rank of the underlying space is at least three, then the group of projectivities of the vector space is essentially the same as the group of automorphisms of the ring, the collineations corresponding to the inner automorphisms of the ring ([4], p. 187).

We shall determine necessary and sufficient conditions that an abstract ring be isomorphic to a ring $T(F, A)$, the set of all linear transformations of the vector space A over the (not necessarily commutative) field F. We also characterize the ring $T_\nu(F, A)$, which is the set of all linear transformations of A of rank $< \aleph_\nu$ over F (where $\aleph_\nu$ is some infinite cardinal). In particular, $T_0(F, A)$ the ring of all linear transformations of A which are of finite rank is characterized as a simple ring with minimal right ideals satisfying a certain annihilation condition on left ideals (Theorem 6.2).

* Received May 1, 1952.

If K is an abstract ring, a right annulet of K is an ideal which is the totality of right annihilators of a subset of K. Left annulets are defined similarly. Our main result (Theorem 7.5) may be stated as follows: *K is isomorphic to the ring $T(F, A)$ of all linear transformations of a vector space A over a field F if and only if*

(1) *K_0, the socle of K, is not a zero ring, and is contained in every non-zero two-sided ideal of K.*

(2) *If J is a left ideal of K which is annihilated on the right only by zero, then $J \geqq K_0$.*

(3) *The sum of two right (left) annulets is a right (left) annulet.*

(4) *K possesses an identity element.*

The main tool in the investigation is an extension of the Galois correspondence developed by Baer in [2], [4] between the annulets in the transformation rings and the subspaces of the underlying vector space. The main result in this direction is Theorem 2.8.

In examining the structure of the rings $T_\nu(F, A)$ an essential result (Theorem 5.1) is the fact that these rings are generated by their idempotents. Our characterization of $T(F, A)$ makes use of the characterization of the rings $T_\nu(F, A)$ although it is clear that the theorem may be proved directly. We prove in Theorem 8.2 that the structure of $T(F, A)$ is completely determined by its ideal theory.

1. Basic concepts. A *linear manifold* is a pair (F, A) consisting of an additive abelian group A and a (not necessarily commutative) field F, in which the field elements operate from the left on the elements of A in the obvious manner.

As usual, a *basis* of the linear manifold (F, A) is a maximal set of linearly independent elements of A. It is well known ([4], p. 14) that every linear manifold possesses a basis, and that the cardinal number of each basis is the same. The *rank* of the linear manifold (F, A) is the cardinal number of any basis of A.

A *linear submanifold* or *subspace* of A is a non-vacuous subset S of A satisfying $S + S = S$ and $FS = S$. To every subspace S of A there exists a subspace Q of A such that $S \cap Q = 0$ and $S + Q = A$ ([4], p. 12). We write $A = S \oplus Q$ in this case, and call Q a *complement* of S.

We recall that an endomorphism of an additive group G is a single-valued mapping of G into itself which preserves addition. A *linear transformation*

or *F-endomorphism* of the linear manifold (F, A) is an endomorphism of A which commutes with the elements of the field F. Thus a linear transformation σ of (F, A) satisfies $(a' + a'')\sigma = a'\sigma + a''\sigma$ for a', a'' in A and $(fa)\sigma = f(a\sigma)$ for a in A and f in F. The totality of linear transformations of the linear manifold (F, A) is denoted by $T(F, A)$. Defining addition and multiplication of linear transformations in the usual way, it is clear that $T = T(F, A)$ is a ring, the ring of all linear transformations of the group A over F. If σ is a linear transformation of (F, A) by its *rank* $r(\sigma)$ is meant the rank of the subspace $A\sigma$. If $\aleph_\nu$ is an infinite cardinal number, we denote by $T_\nu = T_\nu(F, A)$ the totality of elements σ in $T(F, A)$ which satisfy $r(\sigma) < \aleph_\nu$. For each ordinal ν, $T_\nu(F, A)$ is a two-sided ideal of $T(F, A)$. If J is a non-zero two-sided ideal of $T(F, A)$ then $J = T_\mu(F, A)$ for some ordinal μ ([4], p. 198). The same proof as given in [4] can be used to show that every non-zero two-sided ideal of the ring $T_\nu(F, A)$ is of the form $T_\mu(F, A)$ with $\mu \leqq \nu$.

We note that $T_0(F, A)$ consists of all linear transformations of A which are of finite rank, and that $T_0(F, A)$ is contained in every non-zero two-sided ideal of $T_\nu(F, A)$.

2. The Galois theory. In this section, the subspaces of (F, A) are related to certain classes of ideals in the rings of linear transformations. The methods follow those used by Baer in [2] and [4] for the particular ring $T(F, A)$. In some cases the proof may be the same but will be included for the sake of completeness.

Now let K be an arbitrary ring. If S is any subset of K then the totality $\mathfrak{R}(S)$ of elements x in K such that $Sx = 0$ is clearly a right ideal; and such a right ideal we term a *right annulet.* Likewise we denote by $\mathfrak{L}(S)$ the totality of elements y in K such that $yS = 0$. Clearly $\mathfrak{L}(S)$ is a left ideal; and such a left ideal we call a *left annulet.* We have the following:

LEMMA 2.1. *Let K be an arbitrary ring. Then every right annulet $J = \mathfrak{R}[\mathfrak{L}(J)]$ and every left annulet $H = \mathfrak{L}[\mathfrak{R}(H)]$.*

Proof. If H is a left annulet then $H = \mathfrak{L}(Q)$ for some subset Q of K. But $\mathfrak{L}[\mathfrak{R}(\mathfrak{L}(Q))] = \mathfrak{L}(Q)$ holds in any ring, and thus $H = \mathfrak{L}[\mathfrak{R}(H)]$. A similar proof holds for the right annulets.

Now let $E = E(F, A)$ denote any ring of linear transformations of the linear manifold (F, A). If S is a subset of A, then $R(S)$ is the totality of elements σ in E such that $S\sigma = 0$, and $L(S)$ is the totality of elements τ in E such that $A\tau \leqq S$. If S is actually a subspace, then $R(S)$ is a right

ideal and $L(S)$ is a left ideal, the annihilator of S, and the retraction on S respectively.

If J is a subset of E then $N(J)$ is the totality of elements x in A such that $xJ = 0$. By AJ is meant the set of elements aj for a in A and j in J. Although $N(J)$ is always a subspace of A, in general AJ will not be a subspace.

LEMMA 2.2. *Let $E(F, A)$ be any ring of linear transformations of A. Then we have:*

(i) $L(S)R(S) = 0$ *for every subset S of A.*

(ii) $L[N(J)] = \mathfrak{L}(J)$ *for every subset J of E.*

(iii) $R(AJ) = \mathfrak{R}(J)$ *for every subset J of E.*

Proof. (i) $AL(S) \leqq S$ by definition of $L(S)$, and $SR(S) = 0$ by definition of $R(S)$.

Hence $[AL(S)]R(S) = 0$, $A[L(S)R(S)] = 0$ and $L(S)R(S) = 0$.

(ii) is a consequence of the equivalence of the following statements:

$$\sigma \,\varepsilon\, \mathfrak{L}(J), \quad \sigma J = 0, \quad A\sigma J = 0, \quad A\sigma \leqq N(J), \quad \sigma \,\varepsilon\, L[N(J)].$$

(iii) follows from the equivalence of:

$$\sigma \,\varepsilon\, \mathfrak{R}(J), \quad J\sigma = 0, \quad AJ\sigma = 0, \quad \sigma \,\varepsilon\, R(AJ).$$

COROLLARY 2.3. *If $E(F, A)$ is a ring of linear transformations and $AL(S) = S$ holds for a subspace S, then $R(S) = \mathfrak{R}[L(S)]$.*

Proof. By hypothesis $R(S) = R[AL(S)] = \mathfrak{R}[L(S)]$ by Lemma 2.2.

COROLLARY 2.4. *If $E(F, A)$ is a ring of linear transformations and $N[R(S)] = S$ holds for a subspace S, then $L(S) = \mathfrak{L}[R(S)]$.*

Proof. Since $S = N[R(S)]$ we have $L(S) = L[N(R(S))] = \mathfrak{L}[R(S)]$ by Lemma 2.2.

The ring $E(F, A)$ is called a dense ring of linear transformations if given any finite set of elements a_i $(i = 1, 2, \cdots, k)$ in A and linearly independent over F, and any set b_i $(i = 1, 2, \cdots, k)$ there exists a linear transformation e in $E(F, A)$ such that $a_i e = b_i$ for each i.

LEMMA 2.5. *Let $E(F, A)$ be a dense ring of linear transformations of A. Then $AL(S) = S$ for every subspace S of A if and only if $E(F, A)$ contains a dense ring of linear transformations of A of finite rank.*

Proof. Assume $E(F, A)$ contains a dense ring of linear transformations of finite rank. By definition of $L(S)$ it follows that $AL(S) \leqq S$. Now let $s \neq 0$ be in S. By assumption there exists a σ in $E(F, A)$ such that $s\sigma = s$ and $A\sigma$ has finite rank. Since $s\sigma = s$, $s \in A\sigma$. Let $s, m_1, \cdots, m_k$ be a basis of $A\sigma$. By density, there exists $\tau \in E$ such that $s\tau = s$ and $m_i\tau = 0$ for $i = 1, 2, \cdots, k$. Now $A\sigma\tau = Fs \leqq S$ and hence $\sigma\tau \in L(S)$. Then $s \in Fs = A\sigma\tau \leqq AL(S)$ and we have $S \leqq AL(S)$. Combined with the previous inequality, this yields $AL(S) = S$.

Now assume $AL(S) = S$ for every subspace S of A. Let S be of finite (positive) rank. If $L(S) = 0$ then $S = AL(S) = 0$ a contradiction. Hence there exists $e \neq 0$ in $L(S)$, and since $Ae \leqq S$, e has finite rank. Since the set of all transformations in $E(F, A)$ which are of finite rank is clearly a two-sided ideal, and since every non-zero two-sided ideal of a dense ring is again a dense ring ([10], p. 313) it follows that $E(F, A)$ contains a dense ring of linear transformations of finite rank.

LEMMA 2.6. *If $E(F, A)$ is a dense ring of linear transformations, then $N[R(S)] = S$ holds for every subspace S of finite rank. The relation holds for every subspace if, and only if, $E(F, A)$ contains all the linear transformations of A which are of finite rank.*

Proof. Since $SR(S) = 0$, it follows that $S \leqq N[R(S)]$ for every subspace S. To prove $N[R(S)] \leqq S$ we need only show that if a is in A but not in S, then a doesn't belong to $N[R(S)]$. Let $A = S \oplus Fa \oplus U$. If S has finite rank, the density of $E(F, A)$ implies that there exists e in $E(F, A)$ satisfying $ae = a$ and $Se = 0$. If $E(F, A)$ contains all the linear transformations of finite rank, and S has infinite rank, the additional stipulation $Ue = 0$ assures the existence of the required e in $E(F, A)$. Since $Se = 0$ it follows that $e \in R(S)$. But $ae \neq 0$ implies $aR(S) \neq 0$ and hence $a \notin N[R(S)]$ which completes the proof that $N[R(S)] = S$.

Now assume $N[R(S)] = S$ for every subspace S of A. Let S be an arbitrary hyperplane of A and assume, by way of contradiction that $R(S) = 0$. Then $S = N[R(S)] = N(0) = A$ a contradiction. Hence $R(S) \neq 0$ for hyperplanes S. Let σ be a linear transformation of finite rank n. Then we may write $A = \sum_{i=1}^{n} Fs_i \oplus N(\sigma)$, where the s_i are linearly independent. Let S_i be the hyperplane $\sum_{j \neq i} Fs_j \oplus N(\sigma)$. Then since $R(S_i) \neq 0$ there exists for $i = 1, 2, \cdots, n$, $\sigma_i \neq 0$ in $E(F, A)$ such that $S_i\sigma_i = 0$. If $s_i\sigma_i = 0$ it would follow $A\sigma_i = 0$ and hence $\sigma_i = 0$. Now by density of $E(F, A)$, there exists $\tau_i \in E(F, A)$ such that $(s_i\sigma_i)\tau_i = s_i\sigma$. Clearly $\sigma = \sum_{i=1}^{n} \sigma_i\tau_i$, and since σ_i, τ_i are

in the ring $E(F, A)$ for each i, it follows that $\sigma \varepsilon E(F, A)$ completing the proof.

LEMMA 2.7. *Let $E(F, A)$ contain a dense ring of linear transformations of finite rank. Then every left annulet $H = L(AH)$ and every right annulet $J = R[N(J)]$.*

Proof. Let H be a left annulet, so that $H = \mathfrak{L}(Q)$ for some subset Q of $E(F, A)$. But $\mathfrak{L}(Q) = L[N(Q)]$ by Lemma 2.2. Hence $AH = AL[N(Q)] = N(Q)$ by Lemma 2.5. Finally $H = L(AH)$.

If J is a right annlet then

$$J = \mathfrak{R}[\mathfrak{L}(J)] = R[A\mathfrak{L}(J)] = R[AL(N(J))] = R[N(J)].$$

We note that under the conditions imposed here on $E(F, A)$, AH is a subspace of A, whenever H is a left annulet.

A *projectivity* is a one-one mapping of one partially ordered set upon another partially ordered set which preserves the order relation.

A *duality* is a one-one mapping of one partially ordered set upon another partially ordered set which inverts the order relation.

If the partially ordered sets under consideration are lattices, it is clear that projectivities will also preserve cross-cuts and joins, while dualities will interchange cross-cuts and joins.

We now state the essential result of this section.

THEOREM 2.8. *Let $E(F, A)$ contain all those linear transformations of A which are of finite rank. Then*

(i) *The correspondences $R(S)$ and $N(J)$ are reciprocal dualities between the subspaces S of A and the right annulets J of E.*

(ii) *The correspondences $L(S)$ and AH are reciprocal projectivities between the subspaces S of A and the left annulets H of E.*

(iii) *The correspondences $\mathfrak{L}(J)$ and $\mathfrak{R}(H)$ are reciprocal dualities between the right annulets J and the left annulets H of E.*

Proof. (i) By Lemmas 2.6 and 2.7 and Corollary 2.3 we have $S = N[R(S)]$ and $J = R[N(J)]$, $R(S) = \mathfrak{R}[L(S)]$. Hence the correspondence is one-one between the class of all right annulets of E and the totality of subspaces of A. If $S \leqq Q$ are subspaces of A then $R(S) \geqq R(Q)$. If $U \leqq V$ are subsets of $E(F, A)$ then $N(U) \geqq N(V)$, and hence the correspondences are dualities.

(ii) Follows similarly from Lemmas 2.5 and 2.7 and Corollary 2.4.

(iii) Follows directly from Lemma 2.1.

Remark 1. If $E(F,A)$ contains all the linear transformations of A which are of finite rank, the totality of left (right) annulets of $E(F,A)$ forms a complete complemented modular lattice which is projectively equivalent (dual) to the projective geometry determined by the linear manifold (F,A).

Remark 2. The intersection of any number of left (right) annulets is again a left (right) annulet. Hence to every set of left (right) annulets there exists a smallest left (right) annulet containing all the annulets in the given set: the *join* of the annulets in the set. Since annulets are ideals this join will always contain the ideal-theoretical sum, but in general it will be larger.

Remark 3. If $E(F,A)$ contains only a dense ring of linear transformations of finite rank then the statements of Theorem 2.8 remain valid for the set of all finite-dimensional subspaces of A and a corresponding subclass of the set of all right (left) annulets of $E(F,A)$. For, a right ideal J is a right annulet if, and only if, $J = R(S)$ for a subspace S of A. Every left annulet has the form $L(S)$ for a subspace S of A, and if S is a subspace of finite rank then the ideal $L(S)$ is always a left annulet.

3. Primitive rings with minimal ideals. An abstract ring K which is isomorphic to a dense ring of linear transformations is called a *primitive* ring. The following is a collection of known results.

THEOREM 3.1. *Let $E(F,A)$ be a dense ring of linear transformations of A. Then E contains minimal right ideals if, and only if, E contains non-zero linear transformations of finite rank. In this case, the sum of all the minimal right ideals coincides with the sum of all the minimal left ideals, and with $E_0(F,A)$ the totality of linear transformations of A of finite rank which are contained in $E(F,A)$. The ring $E_0(F,A)$ is itself a dense ring of linear transformations of A which is a simple ring (not a zero ring) and is a two-sided ideal of $E(F,A)$ which is contained in every non-zero two-sided ideal of E.*

Proof. Every primitive ring has zero Jacobson radical ([10], p. 310). Since the radical contains all nilpotent ideals ([10], p. 304) it follows that a primitive ring contains no nilpotent ideals. Hence in particular E_0 is not a

zero ring. In any ring without nilpotent ideals the sum of all minimal right ideals coincides with the sum of all minimal left ideals ([11], p. 13). The remainder of the theorem is a restatement of Theorems 29 and 30 of [10] and the fact that any two-sided ideal of a dense ring is itself a dense ring.

LEMMA 3.2. *If K is a simple ring containing minimal right ideals, then K is semi-simple if, and only if, it is not a zero ring. In the event K is not a zero ring, it also contains minimal left ideals, and every right (left) ideal is the sum of minimal right (left) ideals.*

Proof. The sum of all minimal right ideals in any ring is a two-sided ideal ([7]). Since K is simple it is equal to the sum of all its minimal right ideals. If K is semi-simple then certainly it is not a zero-ring. Assume now $K^2 \neq 0$ but that $N \neq 0$ where N is the radical of K. Then since K is simple, $N = K$. Since every minimal right ideal is a left annihilator of the radical ([3], p. 565) it follows that $K^2 = KN = 0$ a contradiction. Hence $N = 0$. The fact that K contains minimal left ideals is due to Artin-Whaples ([1], p. 92). Hence K is also equal to the sum of its minimal left ideals. Thus, the lattice of right (left) ideals of K is a complete modular lattice in which the universal bound is the join of points. It follows then from a theorem of Birkhoff ([5], p. 129) that each element of the lattice is a join of points which completes the proof.

THEOREM 3.3. *Let $E(F, A)$ be a dense ring of linear transformations of A which contains minimal ideals, and let $E_0(F, A)$ be the two-sided ideal of linear transformations of finite rank which are contained in $E(F, A)$. Then*

(i) *Every right (left) ideal of $E_0(F, A)$ is a right (left) ideal of $E(F, A)$.*

(ii) *The minimal right (left) ideals of the rings E and E_0 are the same.*

(iii) *The minimal right (left) annulets are the same as the minimal right (left) ideals.*

(iv) *E_0 itself is a right (left) annulet if, and only if, $E_0 = E$.*

Proof. (i) The ring $E_0(F, A)$ contains minimal ideals since it contains non-zero linear transformations of finite rank. It is a simple (non-zero) ring by Theorem 3.1. By Lemma 3.2 every right (left) ideal is the sum of minimal right (left) ideals. Thus the proof of (i) will be complete if every minimal ideal of E_0 is an ideal of E. But in any ring K a minimal right ideal M satisfies either $M^2 = 0$ or $M = eM = eK$ where $e^2 = e$ is in M ([8],

p. 64). Since E_0 is semi-simple by Lemma 3. 2, it contains no nilpotent ideals and $M = eE_0$ with $e^2 = e$ in M. Clearly $eE_0 \leqq eE$. But $e \in E_0$ implies $eE \leqq E_0$ since E_0 is a right ideal of E. Hence $eE = e(eE) \leqq eE_0$. Thus $M = eE_0 = eE$ and is clearly a right ideal of E. The same argument applies as well to the left ideals.

(ii) This is an immediate consequence of (i) and the fact that E_0 is the sum of all the minimal right (left) ideals of E.

(iii) If M is a minimal right ideal then $M = eE$ with e idempotent. Then $M = \mathfrak{R}[E(1-e)]$ where $E(1-e)$ is the set of elements $x - xe$ with x in E. Thus M is a right annulet, and since annulets are ideals, M is a minimal right annulet. Now, let M be a minimal right annulet. Now $ME_0 \leqq M \cap E_0$ and $ME_0 \neq 0$ since if $ME_0 = 0$ we would have $E_0{}^2 = 0$, a contradiction. Thus $M \cap E_0$ is a non-zero right ideal of E_0 and hence by Lemma 3. 2 certainly contains a minimal right ideal M' which by previous remarks is a minimal right annulet. Since $M' \leqq M$ and both are minimal right annulets it follows $M = M'$ and M is a minimal right ideal. Since the same argument holds for left ideals and left annulets, this completes the proof of (iii).

(iv) If $E_0 = E$ then $E = \mathfrak{R}(0) = \mathfrak{L}(0)$ and is both a right and left annulet.

Now assume E_0 is a right annulet then $E_0 = R(S)$ for a subspace S. If $S = 0$, $E_0 = R(0) = E$ and we are finished. Assume therefore $S \neq 0$. Let $a \neq 0$ be in S then $aE_0 = 0$. But E_0 being a dense ring we must have $aE_0 = A$, a contradiction.

If E_0 is a left annulet, then $E_0 = L(S)$ for a subspace S of A. If $S = A$ then $E_0 = L(A) = E$. Assume therefore $S < A$. By density of E_0 we have $AE_0 = A$, but $AE_0 = AL(S) = S < A$ a contradiction. This completes the proof of the theorem.

Remark 1. Since every non-zero two-sided ideal I of $E(F, A)$ contains $E_0(F, A)$ it is clear that I is an annulet if, and only if, $I = E(F, A)$.

Remark 2. It is easy to see that an identity element of the ring E or E_0 if it exists acts as an identity transformation. Hence E_0 possesses an identity element if, and only if, the rank of A over F is finite.

Remark 3. If $E(F, A)$ is a dense ring and A has finite rank n, then $E(F, A)$ is the ring of all linear transformations of the linear manifold (F, A) or what is essentially the same thing, the ring of all n by n matrices with elements in the field F.

The situation as regards the relation of maximal ideals and maximal annulets is quite different than that of minimal ones. In fact we have the following:

THEOREM 3.4. *The following conditions on a ring K are equivalent:*

(1) *K is a primitive ring with an identity such that every maximal right ideal is an annulet.*

(2) *K is a primitive ring containing minimal ideals and an identity such that every maximal left ideal is an annulet.*

(3) *K is a primitive ring with minimal ideals and an identity such that the product of two right annulets is a right annulet.*

(4) *K is a primitive ring containing minimal ideals and an identity such that the product of two left annulets is a left annulet.*

(5) *K is a primitive ring with minimal ideals and an identity such that the sum of any set of right annulets is a right annulet.*

(6) *K is a primitive ring with minimal ideals and an identity such that the sum of any set of left annulets is a left annulet.*

(7) *K is a primitive ring in which every right and left ideal is an annulet.*

(8) *K is a primitive ring with an identity in which every left ideal is an annulet.*

(9) *K is (for some integer n) the ring of all n by n matrices over a field.*

Proof. It is clear that (9) implies all the other conditions since a total matrix ring is primitive, contains an identity and possesses minimal ideals since it satisfies the minimum condition on right (left) ideals. In fact, every ideal is generated by an idempotent and hence is an annulet ([8], p. 65). The fact that (7) implies (9) is a theorem of Kaplansky ([14], p. 694).

We shall show that each of the other conditions also implies (9).

Assume (1), then every maximal right ideal has the form $R(S)$ for S a subspace of rank 1. Also $N[R(S)] = S$ holds. Thus $L(S) = L[N(R(S))] = \mathfrak{L}[R(S)]$ by Lemma 2.2. If $L(S) = 0$ we have $\mathfrak{L}[R(S)] = 0$. But $R(S) = \mathfrak{R}[\mathfrak{L}(R(S))] = E(F, A)$ by Lemma 2.1, since $R(S)$ is an annulet. But this is impossible since $R(S)$ is a maximal ideal. Hence $L(S) \neq 0$ and

$E(F, A)$ contains non-zero linear transformations of finite rank and $E_0(F, A)$ is a dense ring. If $E_0 < E$ we may imbed (because of existence of an identity) E_0 in a maximal right ideal which is by assumption an annulet. From Remark 1 following Theorem 3.4, it follows $E_0 = E$ and from Remark 3 the conclusion follows.

Assume (2). It follows immediately that every maximal left ideal has the form $L(S)$ for S a hyperplane in A. Again if $E_0 < E$ we imbed E in a maximal left ideal $L(S)$ and the conclusion again follows.

Assume (3), and let S be a hyperplane in A. Now $R(0)R(S) = ER(S)$ is a two-sided ideal of $E \neq 0$, and a right annulet, whence the conclusion follows.

Assume (4), and let S be a subspace of rank 1, then $L(S)$ is an annulet and $L(S)L(A) = L(S)E$ is a two-sided ideal $\neq 0$ and is a left annulet from which the conclusion follows.

Assume (5) or (6). Since E_0 is the sum of all minimal right (left) annulets it follows that E_0 is an annulet from which the conclusion follows.

Assume (8), then K must be a simple ring. For assume $Q \neq 0$ is a two-sided ideal of K. Then Q is a dense ring ([10], Theorem 22). Since Q is a left ideal, it is an annulet. Let $Q = \mathfrak{L}(M) = L(S)$ where $S = N(M)$ (Lemma 2.2). If $S = A$, $Q = L(A) = K$. Hence assume $S < A$. Then $AL(S) \leqq S < A$. But since $L(S)$ is a dense ring $AL(S) = A$. This contradiction shows $Q = 0$ and hence K is simple. It follows from a theorem of Kaplansky ([15], p. 25) that K consists only of transformations of finite rank. Since K contains an identity the theorem follows.

This completes the proof of Theorem 3.4.

In the remainder of this section the ring $E(F, A)$ will be assumed to contain all the linear transformations of A which are of finite rank (that is, $E_0(F, A) = T_0(F, A)$ in our notation).

THEOREM 3.5. *Let $E(F, A)$ contain all the linear transformations of A which have finite rank. Then a left ideal J satisfies $\mathfrak{R}(J) = 0$ if, and only if, J contains all the linear transformations of A which are of finite rank.*

Proof. Assume $J \geqq T_0(F, A) = E_0(F, A)$. Then J is certainly a dense ring, and hence $AJ = A$. The fact that $\mathfrak{R}(J) = 0$ follows from the equivalence of the following statements:

$$\sigma \in \mathfrak{R}(J), \quad J\sigma = 0, \quad AJ\sigma = 0, \quad A\sigma = 0, \quad \sigma = 0.$$

Now assume $\mathfrak{R}(J) = 0$. Let $B \leqq A$ be the subspace spanned by the set of elements AJ. If $B < A$, there exists $\sigma \neq 0$ in $E(F, A)$ satisfying $B\sigma = 0$, $A\sigma \neq 0$. Hence $(AJ)\sigma \leqq B\sigma = 0$.

The following conditions however are equivalent:

$$(AJ)\sigma = 0, \quad A(J\sigma) = 0, \quad J\sigma = 0, \quad \sigma = 0, \text{ a contradiction.}$$

Hence $\{AJ\} = A$. Let $a \neq 0$ be in A. Then there must exist

$$a_1, a_2, \cdots, a_n \text{ in } A \text{ and } j_1, j_2, \cdots, j_n \text{ in } J \text{ such that } a = \sum_{i=1}^{n} a_i j_i.$$

There exist linear transformations of A of finite rank σ_i $(i = 1, 2, \cdots, n)$ in $E_0(F, A)$ such that $a\sigma_i = a_i$. Now $\sigma_i j_i \varepsilon J$ for $i = 1, 2, \cdots, n$ since J is a left ideal. Let $\tau = \sum_{i=1}^{n} \sigma_i j_i$, and $\tau \varepsilon J$. Thus $a\tau = a(\sum_{i=1}^{n} \sigma_i j_i) = \sum_{i=1}^{n} a_i j_i = a$. Now let μ be any linear transformation of rank 1, and let

$$A = Fa \oplus N(\mu) \text{ where } a\mu = b \neq 0.$$

By the above result, there exists α in J such that $b\alpha = b$. Since J is a left ideal $\mu\alpha \varepsilon J$. Thus $a(\mu\alpha) = b\alpha = b$, $N(\mu)\mu\alpha = 0$.

Hence $\mu = \mu\alpha \varepsilon J$. But as in the proof of Lemma 2.6 every finite transformation is the sum of transformations of rank 1, and therefore J contains all finite transformations.

Remark 1. If $E_0(F, A)$ is merely a dense ring then $J \geqq E_0$ implies $\mathfrak{R}(J) = 0$ but not conversely. For, we shall show that this condition assures that $E_0(F, A) = T_0(F, A)$ and there exist many examples of dense rings of linear transformations of finite rank which do not include all the linear transformations of finite rank.

THEOREM 3.6. *Let $E(F, A)$ contain all linear transformations of A which are of finite rank, and let M be a minimal right ideal of E. Then if I is the cross-cut of M and a left ideal H of $E(F, A)$, there exists a unique left annulet H^* such that $I = M \cap H^*$.*

Proof. We shall not give the proof in detail as it is essentially that given in [2] (Theorem 9.1) for the ring $T(F, A)$. It is shown there that AI is a subspace of A and $H^* = L(AI)$ is the required left annulet. We have already shown in Theorem 2.8 that the necessary properties of annulets hold in $E(F, A)$ since $E_0(F, A) = T_0(F, A)$. The modification necessary to take care of the fact that $E(F, A)$ need not possess an identity element is clear.

4. Ideal-theoretic properties of the annulets. In this section we shall examine the structure of annulets in the rings $T_\nu(F,A)$ and relate them to idempotents in the ring. In addition it will be shown that the rings $T_\nu(F,A)$ are generated by their idempotents.

Before proceeding we need the concept of *rank* for elements of a lattice. Let M be a lattice with zero element 0, and let x be an arbitrary element of $M \neq 0$. By *a chain between* 0 *and* x will be meant a well ordered (by the inclusion relation in the lattice) set of distinct elements of M which are bounded above by x, which includes 0, but not x. The *rank of* x is the least upper bound of the cardinal numbers of all chains between 0 and x, if $x \neq 0$. We define the rank of the zero element to be zero.

The totality of right (left) annulets of an arbitrary ring forms a lattice, and hence when we speak of the rank of a right (left) annulet we shall mean its lattice rank as defined above.

Let us consider the lattice of subspaces of a vector space V. If $x \neq 0$ is a subspace of V, consider any chain between 0 and x. If y, z are any elements of this chain and $y < z$, a basis of the subspace y may be extended to a basis of z. If we extend by one basis element at a time, and repeat this procedure for all such y and z, we may in this manner refine the given chain to a densest possible chain. The cardinal number of one of these chains, however, is clearly just the vector space rank of the subspace x. Hence we have shown that for the lattice of subspaces of a vector space the concept of lattice rank of a subspace coincides with its usual vector space rank.

Now let $E(F,A)$ be a dense ring of linear transformations containing all linear transformations of A, which are of finite rank. If J is a left annulet of E, there exists a unique subspace S such that $J = L(S)$. By the use of the preceding arguments and the projectivity of Theorem 2.8 it follows that the rank of the left annulet $L(S)$ is just the vector space rank of S. Now let $H = R(S)$ be a right annulet of E. Since the zero element of the lattice of right annulets is $R(A)$, the preceding results imply that the rank of the right annulet $R(S)$ is the ordinary vector space rank of the quotient space A/S.

If J, J' are right (left) annulets of a ring K and $J \cap J' = 0$, $J \cup J' = K$, we shall say that J and J' are *complementary right* (*left*) *annulets* and either shall be called a *complement* of the other.

THEOREM 4.1. (a) *If J is a right* (*left*) *annulet of $T_\nu(F,A)$ of rank $< \aleph_\nu$ and J' is a complementary right* (*left*) *annulet, then there exists an idempotent e in $T(F,A)$ such that*

$$J = eT_\nu, \quad J' = (1-e)T_\nu \quad (resp.\ J = T_\nu e, \quad J' = (1-e)T_\nu).$$

(b) *A right (left) annulet of* $T_\nu(F, A)$ *is generated by an idempotent if, and only if, it is of rank* $< \aleph_\nu$.

Proof. (a) Let H be a left annulet of rank $< \aleph_\nu$, and H' a complementary left annulet. Then it follows from Theorem 2.8 and our discussion of rank, that $H = L(U)$, $H' = L(W)$ where $A = U \oplus W$ and $r(U) < \aleph_\nu$. Define e as follows: $ue = u$ if $u \varepsilon U$, $We = 0$, so that $N(e) = W$. Then e is idempotent and $Ae = U$ tells us that $e \varepsilon T_\nu(F, A)$ since $r(e) < \aleph_\nu$. Since $Ae = U$ we have $e \varepsilon L(U) = H$. Since H is a left ideal $T_\nu e \leqq H$.

Now let $\tau \varepsilon H = L(U)$ so that $A\tau \leqq U$. Let a be in A. Then $a\tau = (a\tau)e$ since $a\tau \varepsilon U$ and $ue = u$ if $u \varepsilon U$. This implies $\tau = \tau e$ and thus $H \leqq T_\nu e$ which combined with previous inequality gives $H = T_\nu e$ with e idempotent. Now the following statements are equivalent:

$$x \varepsilon T_\nu(1-e), \quad xe = 0, \quad Axe = 0, \quad Ax \leqq Ne = W, \quad x \varepsilon L(W).$$

Hence $H' = T_\nu(1-e)$.

Let J be a right annulet of rank $< \aleph_\nu$, where J' is a complementary right annulet. Then it follows from Theorem 2.8 and our discussion of rank that $J = R(S)$, $J' = R(Q)$ where $A = S \oplus Q$ and $r(A/S) = r(Q) < \aleph_\nu$. Define the linear transformation e as follows: $Se = 0$, $qe = q$ if $q \varepsilon Q$, so that $S = N(e)$. Then e is idempotent and $Ae = Q$ so that $e \varepsilon T_\nu(F, A)$. Since $e \varepsilon R(S) = J$ which is a right ideal we have $eT_\nu \leqq R(S)$. Now let $f \varepsilon R(S)$ so that $Sf = N(e)f = 0$. If $a \varepsilon A$ then $a = ae + (a - ae)$ where $(a - ae)e = 0$ implies $a - ae \varepsilon S$ and therefore $(a - ae)f = 0$. Then $af = aef$ for each a in A implies $f = ef$ so that $R(S) \leqq eT_\nu$. Combined with the previous inequality, we have $J = eT_\nu$, with e idempotent. Now the following are equivalent:

$$x \varepsilon (1-e)T_\nu, \quad ex = 0, \quad Aex = 0, \quad Qx = 0, \quad x \varepsilon R(Q).$$

Hence $J' = (1-e)T_\nu$. This completes the proof of (a).

(b) By Theorem 2.8, every right (left) annulet of $T_\nu(F, A)$ possesses a complementary right (left) annulet. Hence we have already shown that every right (left) annulet of rank $< \aleph_\nu$ is generated by an idempotent element. Assume now H is a left annulet generated by an idempotent $H = L(U) = T_\nu e$ where $e^2 = e \neq 0$ is in T. If $a \neq 0$ and $a \varepsilon A$, we have $aT_\nu = A$ since T_ν is a dense ring. Then certainly $AT_\nu = A$. Now $U = AL(U)$ implies $U = AT_\nu e = Ae$. Since $e \varepsilon T_\nu$, $r(U) = r(Ae) < \aleph_\nu$ and by definition of rank, H is a left annulet of rank $< \aleph_\nu$. Now let J be a right annulet generated by an idempotent e so that $J = eT_\nu$. By Theorem 2.8, $J = R(S)$, S a subspace

of A. Since $N[R(S)] = S$ we have $S = N(eT_\nu)$. Clearly $N(e) = N(eT_\nu)$. We have $A = Ae \oplus Ne$ since e is idempotent, and thus $r(Ae) = r[A/N(e)]$. But since $e \varepsilon T_\nu(F, A)$, $r(Ae) < \aleph_\nu$ and hence $r[A/N(e)] = r(A/S) < \aleph_\nu$. By our discussion of rank for right annulets, we have that J is a right annulet of rank $< \aleph_\nu$. This completes the proof of (b).

Remark 1. It is clear from the proof that Theorem 4.1 is also valid for any ring $E(F, A)$ which contains the ring $T_\nu(F, A)$.

COROLLARY 4.2. *An ideal in $T(F, A)$ is an annulet if, and only if, it is generated by an idempotent.*

Proof. If $r(A) < \aleph_\nu$ then $T(F, A) = T_\sigma(F, A)$ for all ordinals $\sigma \geqq \nu$, hence every annulet is generated by an idempotent. We have previously remarked that in any ring, ideals which are generated by idempotents are certainly annulets. This completes the proof.

We note that Corollary 4.2 is proven in [4].

COROLLARY 4.3. *If J is a right (left) annulet of rank $< \aleph_\nu$ in $T_\nu(F, A)$ and J' is a complementary right (left) annulet, then T is the direct sum of the right (left) ideals J and J'.*

If J, J' are complementary right (left) annulets of $T(F, A)$, then T is the direct sum of the right (left) ideals J and J'.

A ring K is called *regular* if for every a in K, there exists an x in K such that $a = axa$. Such rings were introduced by von Neumann in [16] where existence of an identity was also assumed. However the following statements which are proved there are easily seen to be true without the existence of an identity element.

(1) In a regular ring, every principal right (left) ideal is generated by an idempotent element.

(2) In a regular ring, the sum of two principal right (left) ideals is a principal right (left) ideal.

THEOREM 4.4. *For each ordinal ν we have:*

(i) *The ring $T_\nu(F, A)$ is a regular ring.*

(ii) *The sum of two (and hence a finite number) of right (left) annulets of rank $< \aleph_\nu$ is a right (left) annulet of rank $< \aleph_\nu$.*

(iii) *Every right or left ideal which is finitely generated (and hence every principal ideal) is an annulet and is generated by an idempotent element.*

Proof. (i) The fact that $T(F, A)$ is a regular ring has been noted by Baer ([4], p. 179) and Johnson and Kiokemeister ([13], p. 407). Now let $e \varepsilon T_\nu$, then $e = efe$ for some f in $T(F, A)$. Then $e = (efe)fe = e(fef)e$ and fef is in T_ν since T_ν is a two-sided ideal of T. Hence T_ν is a regular ring. This last trick has been noted by Brown and McCoy ([6], p. 165).

(ii) An annulet of rank $< \aleph_\nu$ is generated by an idempotent (Theorem 4. 1) and is thus principal. Sum of two principal ideals in a regular ring is principal and generated by an idempotent and thus is an annulet of rank $< \aleph_\nu$ again by Theorem 4. 1.

(iii) A finitely generated ideal is a sum of finitely many principal ideals and is principal, since T_ν is a regular ring. Every principal ideal is generated by an idempotent and is therefore an annulet.

COROLLARY 4. 5. (Baer) *An ideal in $T(F, A)$ is an annulet if and only if it is finitely generated, and the sum of a finite number of left (right) annulets in $T(F, A)$ is a left (right) annulet.*

Proof. This is a consequence of Theorem 4. 4, and Corollary 4. 2.

Remark 1. If $T_\nu(F, A) \neq T(F, A)$ then the ring $T_\nu(F, A)$ possesses annulets which are not finitely generated. For T_ν itself is both a right and left annulet and if it were finitely generated it would be generated by an idempotent (Theorem 4. 4). This is impossible since such an idempotent would be an identity element for the ring $T_\nu(F, A)$.

Remark 2. In the rings $T_\nu(F, A)$ every ideal is the sum of annulets, since ideals are always sums of principal ideals. (Sums will, in general, be infinite).

5. The idempotents. We shall show in this section that the rings $T_\nu(F, A)$ are generated by their idempotents. The theorem we obtain is the following:

THEOREM 5. 1. *Let (F, A) be a linear manifold of rank at least two. If $E(F, A)$ is a ring of linear transformations which contains all idempotent transformations of A, of rank $< \aleph_\nu$, then $E(F, A)$ contains the ring $T_\nu(F, A)$. In particular if $E(F, A)$ contains all idempotent transformations of A, then $E(F, A) = T(F, A)$.*

The restriction to linear manifolds of rank at least two is essential, since it is clear that if (F, A) has rank one, the theorem fails to be true.

The proof will be given by proving three lemmas, each of which is a special case of the theorem.

LEMMA A. *Let $A = \sum_\nu Fa_\nu \oplus N(\sigma)$ where the a_ν are linearly independent, and the rank of $N(\sigma)$ is not less than that of $\sum_\nu Fa_\nu$, then σ belongs to the ring generated by those idempotents whose rank does not exceed that of σ.*

Proof. Let $a_\nu\sigma = b_\nu$. If the b_ν were dependent, then:

$$\sum_{i=1}^m f_ib_i = 0, \qquad \sum_{i=1}^m f_i(a_i\sigma) = 0, \qquad (\sum_{i=1}^m f_ia_i)\sigma = 0,$$

$$\sum_{i=1}^m f_ia_i \leqq \sum_\nu Fa_\nu \cap N(\sigma), \qquad \sum_{i=1}^m f_ia_i = 0$$

contradicting independence of the a_i. Since the mapping of a_ν onto b_ν is one to one it is clear that $\sum_\nu Fa_\nu$, and $\sum_\nu Fb_\nu$ have the same rank.

Define α as follows: $a_\nu\alpha = a_\nu$ for all ν, and $N(\sigma)\alpha = 0$. Then α is idempotent and has the same rank as σ.

We let $b_\nu = p_\nu + n_\nu$ where p_ν is in $\sum Fa_\nu$, n_ν is in $N(\sigma)$, and we shall map the a_ν firstly onto the p_ν. Since $r[N(\sigma)] \geqq r(\sum_\nu Fa_\nu)$ it follows that $A = \sum_\nu Fa_\nu \oplus \sum_\nu Fk_\nu \oplus W$ where $r(\sum Fk_\nu) = r(\sum Fa_\nu)$, the k_ν are linearly independent and each k_ν is in $N(\sigma)$, and $W \leqq N(\sigma)$. Since $\sum_\nu Fa_\nu \cap \sum_\nu Fk_\nu = 0$ it follows that $\sum_\nu F(a_\nu - k_\nu) \cap \sum_\nu Fk_\nu = 0$, and we may write

$$A = \sum_\nu F(a_\nu - k_\nu) \oplus \sum_\nu Fk_\nu \oplus W.$$

Define β as follows: $(a_\nu - k_\nu)\beta = 0$, $k_\nu\beta = k_\nu$, $W\beta = 0$. Then $a_\nu\beta = k_\nu$, β is idempotent, and rank β = rank σ. Now since $\sum_\nu Fk_\nu \cap \sum_\nu Fp_\nu = 0$ we have also $\sum_\nu F(k_\nu - p_\nu) \cap \sum_\nu Fp_\nu = 0$. Hence we may write

$$A = \sum_\nu F(k_\nu - p_\nu) \oplus \sum_\nu Fp_\nu \oplus U$$

and define γ as follows: $(k_\nu - p_\nu)\gamma = 0$, $p_\nu\gamma = p_\nu$, $U\gamma = 0$. Then $k_\nu\gamma = p_\nu$, γ is idempotent, and rank $\gamma \leqq$ rank σ since $p_\nu \leqq \sum_\nu Fa_\nu$. Let $\alpha\beta\gamma = \tau$. Then $a_\nu\tau = p_\nu$, and $N(\sigma)\tau = 0$. We next must find a linear transformation ω which maps the a_ν onto the n_ν. Clearly $\sum_\nu Fa_\nu \cap \sum_\nu Fn_\nu = 0$ and therefore

also $\sum_\nu F(a_\nu - n_\nu) \cap \sum_\nu Fn_\nu = 0$. Hence we may write

$$A = \sum F(a_\nu - n_\nu) \oplus \sum Fn_\nu \oplus V$$

and define δ as follows: $(a_\nu - n_\nu)\delta = 0$, $n_\nu\delta = n_\nu$, $V\delta = 0$. Thus $a_\nu\delta = n_\nu$, δ is idempotent and rank $\delta \leqq$ rank σ since the n_ν need not be linearly independent while the b_ν are. Let $\omega = \alpha\delta$, then $a_\nu\omega = n_\nu$, and $N(\sigma)\omega = 0$. Now, $a(\tau + \omega) = p_\nu + n_\nu = b_\nu$, and $N(\sigma)(\tau + \omega) = 0$; therefore $\sigma = \tau + \omega$. This completes the proof, since none of the idempotents used had rank exceeding that of σ.

LEMMA B. *Let σ be a linear transformation of infinite rank. Then σ belongs to the ring of linear transformations generated by all those idempotent transformations whose rank does not exceed that of σ.*

Proof. Let $A = \sum Fa_\nu \oplus N(\sigma)$ where the a_ν are linearly independent and $a_\nu\sigma = b_\nu$. If $r[N(\sigma)] \geqq r(\sum Fa_\nu)$, the conclusion follows from the previous lemma. Hence we may assume $r[N(\sigma)] < r[\sum_\nu Fa_\nu]$. Since $\sum_\nu Fa_\nu$ has infinite rank it follows that $r(\sigma) = r[\sum_\nu Fa_\nu] = r(A)$. Therefore no restrictions are imposed on the ranks of idempotents used.

Now since we have infinite rank we may write $\sum Fa_\nu = \sum Fc_j \oplus \sum Fd_j$ where each subspace has the same rank, and the set consisting of all the c_j and d_j is merely the set of all the a_ν.

Since $r[\sum Fd_j \oplus N(\sigma)] = r(A) \geqq r(\sum_j Fc_j)$, there exists by the previous lemma a transformation α in the ring generated by the idempotents, which satisfies: $c_j\alpha = c_j$, $[\sum Fd_j \oplus N(\sigma)]\alpha = 0$.

In the same manner, since $r[\sum Fc_j \oplus N(\sigma)] \geqq r(\sum_j d_j)$ there exists an appropriate β satisfying: $d_j\beta = d_j$, and $[\sum_j Fc_j \oplus N(\sigma)]\beta = 0$. Clearly $a_\nu(\alpha + \beta) = b_\nu$ for each ν, $N(\sigma)(\alpha + \beta) = 0$ so that $\sigma = \alpha + \beta$, and the proof is complete.

The results of the two preceding lemmas are not directly applicable to linear transformations of finite rank, but by similar arguments we shall prove the following:

LEMMA C. *Let (F, A) be a linear manifold of rank at least two. Then the ring $T_0(F, A)$ coincides with the ring I generated by all the idempotents of finite rank.*

Proof. We show first the following:

(1) If u, v are independent elements of A, there exists a finite idempotent (of rank 1)σ such that $u\sigma = v$.

(2) If $A = Fa \oplus W$, there exists a finite idempotent of (rank 1)τ such that $a\tau = a$ and $W\tau = 0$.

To show (1) merely write $A = Fu \oplus Fv \oplus U$ and define σ by $u\sigma = v$, $v\sigma = v$, $U\sigma = 0$. Then σ is clearly idempotent of rank 1 and has required property. The idempotent τ needed in (2) is uniquely determined by the conditions imposed.

To show that we have all finite transformations in I, it suffices to show that I contains all transformations of rank 1. Thus let $A = Fa \oplus N(\alpha)$, $a\alpha = b$, where α is an arbitrary transformation of rank 1.

From (2) there exists τ such that $a\tau = a$ and $N(\alpha)\tau = 0$. If a, b are independent over F, then there exists by (1) σ such that $a\sigma = b$. Then $a(\tau\sigma) = b$, and $N(\alpha)\tau\sigma = 0$, so that $\alpha = \tau\sigma$. If a, b are dependent, then $b = fa$ where $f \neq 0$ is in F. Since $r(A) \geqq 2$ there exists d in $N(\alpha)$ such that a and d are linearly independent. From what we have already shown there exists an ω_1 in I such that: $a\omega_1 = d$, and $N(\alpha)\omega_1 = 0$. Now applying (1) there exists ω_2 in I such that $d\omega_2 = fa$, since d, fa are independent. Then $a\omega_1\omega_2 = fa$, and $N(\alpha)\omega_1\omega_2 = 0$. This completes the proof of the last lemma, since it is clear that $I \leqq T_0(F, A)$.

Theorem 5.1 now follows immediately from Lemmas B and C.

6. The ring $T_0(F, A)$. In this section we are interested in finding necessary and sufficient conditions that an abstract ring be isomorphic to the ring of all linear transformations which have finite rank.

We recall that a ring $P = P(A)$ of endomorphisms of the additive abelian group A is called *irreducible* if, for every a, not zero in A we have $aP = A$.

If a ring K contains minimal right ideals the *socle* of K is the sum of all its minimal right ideals. If K is without minimal right ideals then its socle is the zero ideal. The socle is always a two-sided ideal. (cf. Dieudonné [7]).

THEOREM 6.1. *Let K be an arbitrary ring. Then there exists a linear manifold (F, A) such that K is isomorphic to a ring $E(F, A)$ of linear transformations of A containing the ring $T_0(F, A)$ if, and only if,* (1) *and at least one of the conditions* (2), (3), *or* (4) *hold.*

(1) *The socle K_0 of K is not a zero ring and is contained in every non-zero two-sided ideal of K.*

(2) *If H is a left ideal of K and $\Re(H) = 0$ then $H \geqq K_0$.*

(3) *If H is a left ideal of K and $\Re(H) = 0$ then H contains a minimal right ideal.*

(4) *If M' is a minimal right ideal and J a left ideal of K then there exists a left annulet J^* such that $M' \cap J = M' \cap J^*$.*

Proof. Assume that K is isomorphic to $E(F, A)$ where $T_0(F, A) \leqq E(F, A) \leqq T(F, A)$. Then (1) holds by virtue of Theorem 3.1 and (2) is a consequence of Theorem 3.5.

Now if (1) and (2) hold then (1) and (3) are valid since (2) implies (3).

Assume now that (1) and (3) are valid. By (1) there exists a minimal right ideal M of K. Let I be the totality of elements x in K such that $Mx = 0$. Assume $I \neq 0$ so that $I \geqq K_0$ by (1) and thus $MK_0 = 0$. Now let I' be the totality of y in K satisfying $yK = 0$. Then we have $0 < M \leqq I'$ so that $I' \neq 0$. It follows from (1), since I' is a two-sided ideal, that $I' \geqq K_0$ or that $K_0{}^2 = 0$ which contradicts (1). Hence, we conclude that $I = 0$, that is $\Re(M) = 0$. In particular $M^2 \neq 0$ so that $M = eK = eM$ where $e^2 = e$ is in M.

For each k in K define an endomorphism σ_k of the additive group M by $x\sigma_k = xk$ for $x \varepsilon M$. The mapping of k onto σ_k constitutes a homomorphism of the ring K onto a ring $P(M)$ of endomorphisms of the group M. If σ_k is the zero endomorphism then $Mk = 0$ and $k = 0$ by preceding remarks. Hence K is actually isomorphic to the ring $P(M)$. We wish to show that $P(M)$ is an irreducible ring of endomorphisms. Let $y \neq 0$ be in M. Then $yK \leqq M$ since M is a right ideal. Since yK is a right ideal we have $yK = 0$ or $yK = M$ by the minimality of M. If $yK = 0$ then $yK_0 = 0$ and the preceding arguments lead again to the contradiction $K_0{}^2 = 0$. Hence $P(M)$ is an irreducible ring of endomorphisms. By Schur's Lemma ([8], p. 57) the totality of endomorphisms of M that commute with the elements of the irreducible ring $P(M)$ is a field F, and by Theorem 6 of [9] the ring of linear transformations $P(F, M)$ is a dense ring. Since $P(M)$ is a ring of right multiplications of M, where M is generated by an idempotent, it follows from [11] Lemma 1 that the commuting field F is isomorphic to $eKe = eMe$. It is easy to see that e is the identity of F and that e acts as an identity operator on the elements of M. Since $Me = (eMe)e = Fe$ the linear transformation σ_e of $P(F, M)$ is of rank 1 so that P contains transformations of finite rank. By Theorem 3.1, $P_0(F, M) = P_0(F, A)$ is a dense ring of

linear transformations of A which are of finite rank. We wish to show that $P_0(F, A)$ contains all linear transformations of A which have finite rank. As in the proof of Lemma 2.6 it is only necessary to show that $P_0(F, A)$ contains all linear transformations of rank 1, since every transformation of rank n is the sum of n transformations of rank 1. Since P is a dense ring, it is sufficient (as in Lemma 2.6) to find a non-zero element σ in P which annihilates an arbitrary hyperplane S of A. Assume by way of contradiction that S is a hyperplane of M, but that $R(S) = 0$. By Corollary 2.3 $R(S) = \mathfrak{R}[L(S)]$. Hence $L(S)$ is a left ideal having right annihilator zero. It follows from (3) that $L(S)$ contains a minimal right ideal M'. By our construction of the ring $P(F, A) = P(F, M)$ we have $L(S)$ is the totality of x in K satisfying $Mx \leqq S$, so that $MM' \leqq S$. But $MM' \neq 0$ since $\mathfrak{R}(M) = 0$, and therefore the minimality of M implies $MM' = M$. Hence $M \leqq S$ which contradicts the fact that S is a hyperplane in M. Hence $R(S) \neq 0$ and we have shown the sufficiency of (1) and (3).

If K is isomorphic to $E(F, A)$ where $T_0(F, A) \leqq E(F, A) \leqq T(F, A)$ then (4) holds by virtue of Theorem 3.6. Assume now (1) and (4). Then by virtue of (1), K is isomorphic to a ring $P(F, A) = P(F, M)$ where $P_0(F, M)$ is a dense ring of linear transformations of finite rank. Let $S < M$ be a hyperplane in M so that $S + S = S$ and $eMeS = MS = S$. Let J be the smallest left ideal of K containing S, so that J is the totality of elements of the form $\sum_{i=1}^{m} (k_i s_i + n_i s_i)$ where $k_i \varepsilon K$, $s_i \varepsilon S$ and n_i is an integer. Since $S < M$ and $S \leqq J$ we have $S \leqq M \cap J$. If $x \varepsilon M \cap J$, then $x = \sum (k_i s_i + n_i s_i)$ and $ex = x$ since $x \varepsilon M = eM = eK$. Hence

$$x = \sum (ek_i s_i + n_i e s_i) = \sum (ek_i s_i + n_i s_i) \varepsilon MS + S = S.$$

Hence we have $M \cap J \leqq S$, and combined with the previous inequality we have $M \cap J = S$. Now by (4), $S = M \cap J^*$ where J^* is a left annulet of K. If $J^* = K$ we have $S = M$ a contradiction. Hence $J^* < K$. Now assume, by way of contradiction, that $R(S) = 0$. Since $S \leqq J^* < K$ we have $\mathfrak{R}(J^*) = \mathfrak{R}(K) = 0$. By Lemma 2.1, $J^* = \mathfrak{L}[\mathfrak{R}(J^*)] = \mathfrak{L}[\mathfrak{R}(K)] = K$ and this contradicts $J^* < K$. Hence $R(S) \neq 0$ and again it follows that $P(eMe, M)$ contains the ring $T_0(eMe, M)$. This completes the proof of Theorem 6.1.

Let $\sigma > \nu$, then if $T_\sigma \neq T_\nu$ the ring $E = T_\sigma / T_\nu$ is a primitive ring containing no minimal ideals ([15], p. 18). Hence E contains no non-zero transformations of finite rank. Since $E_0 = 0$ the ring satisfies (2) but fails to satisfy all of (1) since the socle is a zero ring.

Since the condition (2) seems more desirable than (3) or (4) we shall use this condition in our future characterizations.

We are now in a position to characterize $T_0(F,A)$ itself.

THEOREM 6.2. *Let K be an arbitrary ring. Then there exists a linear manifold (F,A) such that K is isomorphic to the ring $T_0(F,A)$ of all linear transformations of A which are of finite rank if and only if*

(1) *K is a simple ring (not a zero ring) containing minimal right ideals.*

(2) *If H is a left ideal of K and $\mathfrak{R}(H)=0$ then $H=K$.*

Proof. Since K is simple and possesses minimal right ideals we have $K=K_0$ its socle (since the socle is always a two-sided ideal). Hence K satisfies conditions (1) and (2) of Theorem 6.1 and therefore K is isomorphic to $E(F,A)$ where $T_0(F,A) \leqq E(F,A) \leqq T(F,A)$. But $E(F,A)$ in this case is simple, and since $T_0(F,A)$ is always a two-sided ideal we have $E(F,A)=T_0(F,A)$.

Now $T_0(F,A)$ satisfies (2) by Theorem 3.5 and (1) by Theorem 1 of [9]. This completes the proof.

An example of a dense ring of linear transformations of finite rank $P(F,A)$ which satisfies $P(F,A)=P_0(F,A)<T_0(F,A)$ is easily constructed as follows. Let (F,A) be a linear manifold with a countable basis, and let $P(F,A)=P_0(F,A)$ denote the ring of linear transformations which consists of only those transformations which annihilate all but a finite number of the basis elements. Then $P(F,A)$ is certainly a dense ring of transformations of finite rank, but does not, for example, contain the linear transformation of finite rank which maps every basis element into a fixed basis element b. Hence, $P(F,A)$ satisfies (1) but not (2) of the theorem since $P(F,A)<T_0(F,A)$.

7. The rings $T_\nu(F,A)$. In this section we shall characterize the rings $T_\nu(F,A)$ and from this derive a characterization of $T(F,A)$.

As ideals of $T(F,A)$ the rings $T_\nu(F,A)$ are easily characterized abstractly as follows:

THEOREM 7.1. *The ideal $T_\nu(F,A)$ of $T(F,A)$ is the sum of all right annulets of rank $<\aleph_\nu$, and the sum of all left annulets of rank $<\aleph_\nu$.* (For the definition of rank of annulets see the beginning of Section **4**).

Proof. Let $\sigma=\sum_{i=1}^{k}\sigma_i$ where σ_i belongs to a right annulet of rank $<\aleph_\nu$,

that is $\sigma_i \varepsilon R(S_i)$ where $r(A/S_i) < \aleph_\nu$. Now, $A\sigma \leqq \sum_{i=1}^{k} A\sigma_i$, hence $r(A\sigma) \leqq \sum_{i=1}^{k} r(A\sigma_i)$. But $r(A\sigma_i) \leqq r(A/S_i) < \aleph_\nu$, and hence $r(A\sigma) < \aleph_\nu$. Then σ is a linear transformation of rank $< \aleph_\nu$ and $\sigma \varepsilon T_\nu(F, A)$.

Now let $\tau \varepsilon T_\nu(F, A)$ where $A = S \oplus N(\tau)$ and thus $r(A\tau) = r(S\tau) = r(S) < \aleph_\nu$. Then $\tau \varepsilon R[N(\tau)]$ a right annulet of rank $< \aleph_\nu$ and hence certainly belongs to a sum of such right annulets. This completes the proof of the assertion concerning right annulets.

Now consider left annulets, let $\sigma = \sum_{i=1}^{k} \sigma_i$ where $\sigma_i \varepsilon L(S_i)$ and $r(S_i) < \aleph_\nu$. Then $A\sigma_i \leqq S_i$ by definition of $L(S_i)$,

$$A\sigma \leqq \sum_{i=1}^{k} A\sigma_i \leqq \sum_{i=1}^{k} S_i = S$$

where $r(S) < \aleph_\nu$. Thus the rank of σ is less than $\aleph_\nu$ and $\sigma \varepsilon T_\nu(F, A)$. Again if $\tau \varepsilon T_\nu(F, A)$ then $A\tau = S$ where $r(S) < \aleph_\nu$. Thus $\tau \varepsilon L(S)$ a left annulet of rank $< \aleph_\nu$. This completes the proof.

Remark 1. For the special case of $\nu = 0$, we have that $T_0(F, A)$ is both the sum of all the minimal right ideals, and the sum of all the minimal left ideals of $T(F, A)$, which we previously had shown.

If $E(F, A)$ is a ring of linear transformations then the totality of linear transformations of rank $< \aleph_\nu$ (for any infinite cardinal) contained in $E(F, A)$ is a two-sided ideal. The arguments used in the preceding theorem remain valid in $E(F, A)$ if $E_0(F, A) = T_0(F, A)$ by Theorem 2. 8. Hence we have also proved the following:

COROLLARY 7. 2. *Let $E(F, A)$ be a dense ring of linear transformations containing all linear transformations of A which are of finite rank, then the two-sided ideal $E_\nu(F, A)$ of all transformations of rank $< \aleph_\nu$ in $E(F, A)$ coincides with the sum of all right annulets of rank $< \aleph_\nu$ and with the sum of all left annulets of rank $< \aleph_\nu$.*

THEOREM 7. 3. *Let K be an arbitrary ring. Then there exists a linear manifold (F, A) such that K is isomorphic to a ring $E(F, A)$ of linear transformations of A containing the ring $T_\nu(F, A)$ if, and only if, K satisfies conditions* (1), (2), (3) *below.*

(1) *The socle K_0 of K is not a zero ring and is contained in every non-zero two-sided ideal of K.*

(2) *If H is a left ideal of K and $\mathfrak{R}(H) = 0$ then $H \geqq K_0$.*

(3) *If J is a left (right) annulet of K of rank $< \aleph_\nu$ and J' is a left (right) annulet of K complementary to J, then there exists an idempotent e in J such that $J = Ke$, $J' = K(1 - e)$ (resp. $J = eK$, $J' = (1 - e)K$).*

We note again that (3) is not meant to imply the existence of an identity element in K. As far as sufficiency is concerned either the statement regarding right or left annulets is enough. Of course, both statements will be shown to be necessary.

Proof. The necessity of (1) and (2) follows from Theorem 6. 1. Condition (3) is necessary by virtue of Theorem 4. 1 and Remark 1 following that theorem.

Now assume K satisfies (1), (2), and (3). By Theorem 6. 1, K is isomorphic to a ring of linear transformations $E(F, A)$ which contains $T_0(F, A)$. If the rank of A over F is one, then since $E(F, A)$ is a dense ring, it is already the ring of all linear transformations of A of rank $< \aleph_\nu$ for every ordinal ν. Hence without loss of generality we may assume $r(A) \geqq 2$. In order to show that $E(F, A) \geqq T_\nu(F, A)$ it is, by virtue of Theorem 5. 1, only necessary to show that E contains all idempotent linear transformations of rank $< \aleph_\nu$. If e is any idempotent in $T_\nu(F, A)$, we may write $A = Ae \oplus N(e)$ where e is the identity on the subspace Ae of rank $< \aleph_\nu$, and an annihilator of the subspace $N(e)$. Hence to prove $E \geqq T_\nu$ we must show that to each decomposition $A = S \oplus Q$ where $r(S) < \aleph_\nu$, there exists a transformation in $E(F, A)$ which is the identity on S and annihilates Q. To this end, assume $A = S \oplus Q$ where $r(S) < \aleph_\nu$ and assume (3) for left annulets. Then by Theorem 2. 8, $E = L(S) \cup L(Q)$, and $0 = L(S) \cap L(Q)$, where $L(S)$ is a left annulet of rank $< \aleph_\nu$. Then $J = L(S)$ and $J' = L(Q)$ are complementary left annulets satisfying the hypothesis of (3). Hence by (3) $L(S) = Ee$ and $L(Q) = E(1 - e)$, where $e^2 = e \neq 0$. But $S = AL(S) = AEe$ implies that the idempotent e is an identity on S. Now $Q = AL(Q) = AE(1 - e)$ implies that $Qe = 0$ since $(1 - e)e = 0$.

Similarly if we assume (3) for right annulets we have $E = R(S) \cup R(Q)$, and $0 = R(S) \cap R(Q)$, where $R(Q)$ is a right annulet of rank $< \aleph_\nu$, so that by (3) there exists an idempotent e in $R(Q)$ satisfying $R(Q) = eE$ and $R(S) = (1 - e)E$. Since $e \varepsilon R(Q)$ we have $Qe = 0$. Now $SR(S) = 0$ implies $S[(1 - e)E] = 0$ or $[S(1 - e)]E = 0$. Hence $S(1 - e) = 0$, since E annihilates only the zero subspace. Hence if $s \varepsilon S$, $s - se = 0$ or $s = se$, and e is the identity on S. Thus $E(F, A) \geqq T_\nu(F, A)$ which completes the proof.

THEOREM 7.4. *Let K be a ring of linear transformations which contains the ring $T_\nu(F,A)$. Then the following conditions are equivalent:*

(i) $K = T_\nu(F,A)$.

(ii) *K is equal to the sum of all its left (right) annulets of rank $< \aleph_\nu$.*

(iii) *The proper two-sided ideals of K form a well ordered set of order type ν.*

Proof. Assume (i). The ring $T_\nu(F,A)$ satisfies (ii) by virtue of Corollary 7.2 applied to the ring $T_\nu(F,A)$. Now the ordinal number ν represents the order type of the well ordered set of infinite cardinal numbers preceding $\aleph_\nu$. Hence, by the definition of $T_\nu(F,A)$, and the fact that all the non-zero two-sided ideals of T_ν have the form T_σ with $\sigma \leqq \nu$, it follows that ν is also the ordinal number of the well ordered set of proper two-sided ideals of $T_\nu(F,A)$. Hence (iii) is true.

Now assume $K \geqq T_\nu(F,A)$ and (ii) holds. Then Corollary 7.2 implies that K contains no linear transformations of rank $\geqq \aleph_\nu$ or $K = T_\nu(F,A)$. Clearly the assumption that $K > T_\nu(F,A)$ also violates (iii). Hence (iii) also implies that $K = T_\nu(F,A)$. This completes the proof of Theorem 7.4.

Remark 1. It is a consequence of the last theorems that the conditions (1), (2), (3) of Theorem 7.3 and (ii) or (iii) of Theorem 7.4 characterize the ring $T_\nu(F,A)$.

THEOREM 7.5. (MAIN THEOREM). *Let K be an arbitrary ring. Then there exists a linear manifold (F,A) such that K is isomorphic to the ring $T(F,A)$ of all linear transformations of A if and only if*

(1) *The socle K_0 of K is not a zero-ring and is contained in every non-zero two-sided ideal of K.*

(2) *If H is a left ideal of K and $\mathfrak{R}(H) = 0$ then $H \geqq K_0$.*

(3) *The sum of two right (left) annulets is a right (left) annulet.*

(4) *K possesses an identity element.*

Proof. Let J, J' be complementary right (left) annulets so that $J \cap J' = 0$ and $J \cup J' = K$. Since by (3) sums of right (left) annulets are right (left) annulets, it follows that $K = J \oplus J'$. Since K possesses an identity element it follows by a lemma of von Neumann ([16], p. 708) that there exists an idempotent e in K such that $J = eK$, $J' = (1-e)K$ [resp. $J = Ke$, $J' = K(1-e)$]. Hence K satisfies (1), (2), and (3) of Theorem

7.3 for every ordinal ν. Hence by Theorem 7.3, K is isomorphic to $E(F,A)$ where $T_\nu(F,A) \leqq E(F,A) \leqq T(F,A)$ for every ν and thus $E(F,A) = T(F,A)$.

It is clear that $T(F,A)$ possesses an identity element, and satisfies (1) and (2) by Theorem 7.3. The fact that $T(F,A)$ satisfies (3) is a consequence of Corollary 4.5. This completes the proof.

Remark 1. It is clear that the full strength of (3) was not needed. The following is sufficient:

(3′) If J and J' are complementary right (left) annulets of K, then K is the direct sum of the right (left) ideals J and J'.

8. Uniqueness theorems. A *semi-linear transformation* of a linear manifold (F,A) upon a linear manifold (G,B) is a pair $\sigma = (\sigma', \sigma'')$ consisting of an isomorphism σ' of the additive group A upon the additive group B, and an isomorphism σ'' of the field F upon the field G subject to the condition $(fa)^{\sigma'} = f^{\sigma''}a^{\sigma'}$ for f in F and a in A. We note that σ is a linear transformation if $F = G$ and $\sigma'' = 1$. There will be no confusion if in the future we use the same symbol σ both for the isomorphism σ' of A upon B and the isomorphism σ'' of F upon G.

Since $T_\nu(F,A)$ is for every ν, a dense ring containing minimal right ideals, a direct application of a theorem of Jacobson ([10], p. 318) yields the following result:

THEOREM 8.1. *If α is an isomorphism of $T_\nu(F,A)$ upon $T_\nu(G,B)$, then there exists a semi-linear transformation σ of (F,A) upon (G,B) such that $t^\alpha = \sigma^{-1}t\sigma$ for every t belonging to $T_\nu(F,A)$.*

Hence the representations we have obtained are essentially unique.

Let K and K' be abstract rings. A mapping ϕ shall be called *a projection of the ideal theory* of K upon that of K' if ϕ is at the same time a projectivity of the set of right ideals of K upon the set of right ideals of K', the set of left ideals of K upon the set of left ideals of K', and the set of two-sided ideals of K upon those of K' and which also satisfies $(LR)^\phi = L^\phi R^\phi$ if L is any left ideal, and R any right ideal of K. (The statement that J is a left (right) ideal is not meant to preclude the possibility that J is a two-sided ideal).

If ϕ is a projection of the ideal theory of K upon that of K' it follows that ϕ maps right (left) annulets upon right (left) annulets, and in particular the right annihilator of a left ideal onto the right annihilator of its image. If I is a two-sided ideal, setting $L = R = I$ in condition above gives

$(I^2)^\phi = (I^\phi)^2$ so that a two-sided ideal which is not a zero ring is mapped upon a two-sided ideal which is not a zero ring.

If there exists a projection of the ideal theory of K upon the ideal theory of K', we shall say that the *rings K and K' have the same ideal theory.*

THEOREM 8.2. *Assume the linear manifold (F, A) has rank at least three, and that the ring K possesses an identity element. Then the ring K and the ring of linear transformations $T(F, A)$ have the same ideal theory if and only if they are isomorphic. Moreover every projection of the ideal theory of K upon the ideal theory of $T(F, A)$ is induced by a unique ring isomorphism of K upon $T(F, A)$.*

Proof. Isomorphic rings have the same ideal theory. Now assume K and $T(F, A)$ have the same ideal theory. It follows from Theorem 7.5 that $T(F, A)$ satisfies the condition (1), (2), (3) and (4) of that theorem. From our preceding remarks it follows that the ring K also satisfies the same conditions. Hence, Theorem 7.5 then implies that there exists a linear manifold (G, B) so that K is isomorphic to the ring $T(G, B)$ of all linear transformations of B. It now follows from Theorem 2.8 that there exists a projectivity of the system of subspaces of the linear manifold (F, A) upon the totality of left annulets of $T(F, A)$, and a projectivity of the system of subspaces of (G, B) upon the totality of left annulets of $T(G, B)$. Since $T(F, A)$ and $T(G, B)$ have the same ideal theory, there exists a projectivity of the totality of left annulets of $T(G, B)$ upon the totality of left annulets of $T(F, A)$. Thus, there exists a projectivity of (F, A) upon (G, B). Since $r(A) \geqq 3$ it follows from The Fundamental Theorem of Projective Geometry (Baer, [4], p. 44) that this latter projectivity is induced by a semi-linear transformation σ of (F, A) upon (G, B). If $\eta \in T(F, A)$, it is easily verified that the correspondence of η and $\sigma^{-1}\eta\sigma$ is an isomorphism of $T(F, A)$ upon $T(G, B)$ which is isomorphic to K. Now let the product of the projection of $T(F, A)$ upon K by the isomorphism of K upon $T(G, B)$ map the left annulet $L(S)$ of $T(F, A)$ upon the left annulet $L(U)$ of $T(G, B)$. Then the semi-linear transformation σ satisfies $S^\sigma = U$. But the equivalence of the relations:

$$\eta \in L(S), \qquad A\eta \leqq S, \qquad B(\sigma^{-1}\eta\sigma) \leqq U, \qquad \sigma^{-1}\eta\sigma \in L(U)$$

shows that the isomorphism we have constructed has the desired effect on the left annulets. In a similar way it has the desired effect on the right annulets. Since every ideal of $T(F, A)$ is a sum of annulets, (Remark 2 following Corollary 4.5) it follows that the ring isomorphism induces the projection of the ideal theory of $T(F, A)$ upon the ideal theory of K.

Now let α, β be two isomorphisms which induce the same projection of the ideal theory of $T(F, A)$ upon that of K. Then $\alpha\beta^{-1}$ is an automorphism of $T(F, A)$ which, in particular, leaves invariant every left annulet of $T(F, A)$. Hence $\alpha\beta^{-1} = 1$ ([4], p. 187) and $\alpha = \beta$. This completes the proof of the theorem.

Remark 1. If the linear manifold (F, A) has rank less than three, the theorem is no longer valid. Let (F, A) and (G, B) have rank one. Then the rings $T(F, A)$ and $T(G, B)$ are essentially the same as the fields F and G respectively, and hence have the same ideal theory. But the fields F and G need not be isomorphic. For the case of rank two, we may let (F, A) and (G, B) be linear manifolds of rank two which are projectively equivalent, but such that there exists no semi-linear transformation of (F, A) upon (G, B). Examples of such manifolds are given in [4], pp. 50-51. By Theorem 2.8 it follows that there exists a projectivity of the totality of left (right) annulets of the ring $T(F, A)$ upon the left (right) annulets of the ring $T(G, B)$. Since the ranks are finite, every ideal is an annulet. Using the fact that these rings are simple it is easily verified that these projectivities actually constitute a projection of the ideal theory of $T(F, A)$ upon that of $T(G, B)$. The rings $T(F, A)$ and $T(G, B)$ however cannot be isomorphic, since according to Theorem 8.1 such an isomorphism is induced by a semi-linear transformation of (F, A) upon (G, B).

Since the correspondence of η and $\sigma^{-1}\eta\sigma$ of the theorem clearly maps linear transformations of finite rank, upon linear transformations of finite rank we also have the following:

COROLLARY 8.3. *Assume the linear manifold (F, A) has rank at least three. Then every projection of the ideal theory of $T_0(F, A)$ upon the ideal theory of an abstract ring K is induced by a unique ring isomorphism.*[1]

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SYMMETRIC AND ANTI SYMMETRIC KRONECKER SQUARES AND INTERTWINING NUMBERS OF INDUCED REPRESENTATIONS OF FINITE GROUPS.*

By George W. Mackey.

Introduction. This paper is a continuation of part I of an earlier article [4]. There a number of results of Frobenius, Shoda and Artin were unified by deriving them as corollaries of a theorem on the structure of the Kronecker product of two induced representations. In the first half of the present paper we supplement the main theorem of [4] with parallel results on the symmetric and anti symmetric components of the Kronecker square of a single induced representation. These results of course have corollaries about the symmetric and anti symmetric intertwining numbers of pairs of adjoint induced representations. The special cases which deal with the self intertwining numbers of a permutation representation admit simple direct proofs and in the second half of the paper we use the results in these cases and the methods and results of [4] to extend the unification given in [4] so as to encompass certain results of Frame [1], [2] and Wigner [6]. Wigner's results are extended somewhat and, we think, made to seem less mysterious. A reader interested only in the applications in the second half may proceed directly from Section **1** to Theorem 2′ in Section **2**; going back however to read Corollary 2 to Theorem 2 (which is also a corollary to Theorem 2′) and the discussion following this corollary.

As with the results of [4] extensions are possible to Hilbert space representations of locally compact groups. We shall develop these extensions in detail elsewhere and confine ourselves here to a few indicatory remarks in a final paragraph.

We shall assume that the reader is familiar with [4] and shall use terminology and notation introduced therein without further explanation. We shall not in general distinguish between equivalent representations and shall use the term "restriction of a representation" in each of the following contexts: (a) To denote the representation of a subgroup obtained from a representation of a group by ignoring values of the group variable outside of the subgroup. (b) To denote the representation defined by an invariant sub-

* Received May 23, 1952.

space of another representation by ignoring what the linear transformations do outside of this subspace. We shall speak respectively of the restriction to the given subgroup and the restriction to the given subspace. In one case we change the group being represented and in the other the representation space.

1. Symmetry and anti symmetry of Kronecker squares and intertwining operators. Let $\mathfrak{G}$ be a finite group and let $U, x \to U_x$, be an arbitrary representation of $\mathfrak{G}$ by linear transformations in a vector space $\mathfrak{H}(U)$ over a field $\mathfrak{F}$ of odd characteristic. The space of the Kronecker square $U \otimes U$ of U is then the set of all linear transformations T from $\overline{\mathfrak{H}(U)}$ to $\mathfrak{H}(U)$ and $(U \otimes U)_x(T) = U_x T U^*_x$. Hence

$$((U \otimes U)_x(T))^* = (U_x T U^*_x)^* = U_x T^* U^*_x = (U \otimes U)_x(T^*).$$

Thus the involutory linear transformation $T \to T^*$ commutes with $(U \otimes U)_x$ for all x. It follows that the subspaces defined by the equations $T = T^*$ and $T = -T^*$ are invariant and define a two term direct sum decomposition of $U \otimes U$. Thus $U \otimes U = U \circledS U + U \circledA U$ where $U \circledS U$ is obtained by restricting $U \otimes U$ to the set of all $T \varepsilon \mathfrak{H}(U \otimes U)$ with $T = T^*$ and $U \circledA U$ is obtained by restricting $U \otimes U$ to the set all $T \varepsilon \mathfrak{H}(U \otimes U)$ with $T = -T^*$. We shall call $U \circledS U$ and $U \circledA U$ respectively the symmetric and anti symmetric Kronecker squares of U.

An intertwining operator T for $\bar{U}$ and U is a member of $\mathfrak{H}(U \otimes U)$ which is carried into itself by all $(U \otimes U)_x$. It is easy to see that whenever T is an intertwining operator then T^* is also. Thus the space of intertwining operators for $\bar{U}$ and U splits as a direct sum of two subspaces and we have $\mathfrak{J}(\bar{U}, U) = \mathfrak{J}_S(\bar{U}, U) + \mathfrak{J}_A(\bar{U}, U)$ where $\mathfrak{J}_S(\bar{U}, U)$ is the dimension of the space of all intertwining operators T for which $T = T^*$ and $\mathfrak{J}_A(\bar{U}, U)$ is the dimension of the space of all intertwining operators T for which $T = -T^*$. We call $\mathfrak{J}_S(\bar{U}, U)$ and $\mathfrak{J}_A(\bar{U}, U)$ the symmetric and anti symmetric intertwining numbers of $\bar{U}$ and U respectively. It is clear that $\mathfrak{J}_S(\bar{U}, U) = \mathfrak{J}(I, U \circledS U)$ and $\mathfrak{J}_A(\bar{U}, U) = \mathfrak{J}(I, U \circledA U)$ where I is the one dimensional identity representation of $\mathfrak{G}$.

LEMMA 1. *If U and V are arbitrary representations of $\mathfrak{G}$ then*

$$(U + V) \circledS (U + V) = U \circledS U + V \circledS V + U \otimes V$$

and

$$(U + V) \circledA (U + V) = U \circledA U + V \circledA V + U \otimes V$$

where of course $V \otimes U$ could replace its equivalent $U \otimes V$.

Proof. It is obvious that $(U+V)\otimes(U+V)$ has a natural decomposition as a direct sum $U\otimes U+V\otimes V+U\otimes V+V\otimes U$ where $T\rightarrow T^*$ leaves $\mathfrak{H}(U\otimes U)$ and $\mathfrak{H}(V\otimes V)$ invariant and maps $\mathfrak{H}(U\otimes V)$ and $\mathfrak{H}(V\otimes U)$ linearly onto one another. Let $T=T_1, T_2, T_3, T_4$ be an arbitrary element in $\mathfrak{H}(U\otimes U)\oplus\mathfrak{H}(V\otimes V)\oplus\mathfrak{H}(U\otimes V)\oplus\mathfrak{H}(V\otimes U)$. Then $T^*=T^*_1, T^*_2, T^*_4, T^*_3$ so that $T=T^*$ if and only if $T^*_1=T_1$, $T^*_2=T_2$, $T^*_4=T_3$ and $T^*_3=T_4$ and $T=-T^*$ if and only if $T^*_1=-T_1$, $T^*_2=-T_2$, $T^*_3=-T_4$ and $T^*_4=-T_3$. It follows that the space of $(U+V)\circledS(U+V)$ is the set of all $T_1\,T_2,T_3,T^*_3$ where $T_1\,\varepsilon\,\mathfrak{H}(U\circledS U)$, $T_2\,\varepsilon\,\mathfrak{H}(V\circledS V)$ and $T_3\,\varepsilon\,\mathfrak{H}(U\otimes V)$. Thus

$$(U+V)\circledS(U+V)=U\circledS U+V\circledS V+W$$

where W is a representation which is equivalent to $U\otimes V$ *by* way of the mapping $T_3,T^*_3\rightarrow T_3$. $(U+V)$ Ⓐ $(U+V)$ may be treated in parallel fashion.

COROLLARY.

$$\mathfrak{I}_S(\overline{U+V},U+V)=\mathfrak{I}_S(\bar{U},U)+\mathfrak{I}_S(\bar{V},V)+\mathfrak{I}(\bar{U},V)$$

and

$$\mathfrak{I}_A(\overline{U+V},U+V)=\mathfrak{I}_A(\bar{U},U)+\mathfrak{I}_A(\bar{V},V)+\mathfrak{I}(\bar{U},V).$$

Making use of the notation $c(U)=\mathfrak{I}_S(\bar{U},U)-\mathfrak{I}_A(\bar{U},U)$ we have

COROLLARY. $c(U+V)=c(U)+c(V)$.

Note that if U is irreducible and $\mathfrak{F}$ is algebraically closed then $\mathfrak{I}(\bar{U},U)=1$ or 0 according as U is or is not equivalent to $\bar{U}$ and that hence $c(U)=1,\,-1$, or 0. This invariant of irreducible representations was introduced by Frobenius and Schur in [3] in order to classify representations over the complex field according to their "reality." They show that $c(U)=1,\,-1$, or 0 according as U may be realized by real matrices, has a real character function but may not be realized by real matrices or has a non real character function.

LEMMA 2. *If $\mathfrak{F}$ is algebraically closed and U and V are irreducible representations of $\mathfrak{G}_1$ and $\mathfrak{G}_2$ then $c(U\times V)=c(U)c(V)$.*

Proof. Clearly $\overline{U\times V}=\bar{U}\times\bar{V}$. Thus $U\times V=\overline{U\times V}$ if and only if $U=\bar{U}$ and $V=\bar{V}$. Thus $c(U\times V)=0$ if and only if $c(U)=0$ or $c(V)=0$; that is if and only if $c(U)c(V)=0$. If $U=\bar{U}$ and $V=\bar{V}$ let T be an intertwining operator for $\bar{U}$ and U and let S be an intertwining

operator for $\bar{V}$ and V. Then $T \times S$ is an intertwining operator for $\overline{U \times V}$ and $U \times V$. Moreover $(T \times S)^* = T^* \times S^*$. The truth of the lemma is now evident.

COROLLARY. *If $\mathfrak{F}$ is algebraically closed and U and V are direct sums of irreducible representations then $c(U \times V) = c(U)c(V)$.*

LEMMA 3. $c(\bar{U}) = c(U)$ *for all* U.

Proof.

$$c(\bar{U}) = \mathfrak{I}_S(U, \bar{U}) - \mathfrak{I}_A(U, \bar{U}) = \mathfrak{I}(I, \bar{U} \circledS \bar{U}) - \mathfrak{I}(I, \bar{U} \textcircled{A} \bar{U})$$

$$= \mathfrak{I}(I, \overline{U \circledS U}) - \mathfrak{I}(I, \overline{U \textcircled{A} U}) = \mathfrak{I}(I, U \circledS U) - \mathfrak{I}(I, U \textcircled{A} U) = c(U).$$

LEMMA 4.[1] *Let $\mathfrak{F}$ be algebraically closed. Let G be a subgroup of the finite group $\mathfrak{G}$ and let L be an irreducible representation of $\mathfrak{G}$. Suppose that M, the restriction of L to G, is a direct sum of non equivalent irreducible representations M_j. Then $c(M_j) = c(L)$ for all j.*

Proof. It is clear that every intertwining operator T for $\bar{M}$ and M is uniquely a sum of intertwining operators T_j where T_j is zero on $\overline{\mathfrak{H}(M_1)} \oplus \overline{\mathfrak{H}(M_2)} \oplus \cdots \overline{\mathfrak{H}(M_{j-1})} \oplus \overline{\mathfrak{H}(M_{j+1})} \oplus \cdots \overline{\mathfrak{H}(M_n)}$ and has $\mathfrak{H}(M_j)$ for its range. This is so in particular for any intertwining operator which intertwines $\bar{L}$ and L. But such an operator if not zero is non singular. Hence no T_j is zero and we have either $T = T^*$ and hence $T_j = T^*_j$ for all j or we have $T = -T^*$ and hence $T_j = -T^*_j$ for all j.

2. Symmetric and anti symmetric squares of induced representations.

Let G be a subgroup of the finite group $\mathfrak{G}$ and let L be a representation of G. It follows from Theorem 2 of [4] that $U^L \otimes U^L$ is a direct sum over the $G:G$ double cosets of certain other induced representations. In this section we shall show that $U^L \circledS U^L$ and $U^L \textcircled{A} U^L$ are also direct sums of induced representations and describe these representations quite explicitly. We assume of course that the underlying field is of characteristic different from two. To begin with let us recall that via the natural mapping of $\mathfrak{G}$ on the diagonal subgroup $\tilde{\mathfrak{G}}$ of $\mathfrak{G} \times \mathfrak{G}$ $U^L \otimes U^L$ is the restriction to $\tilde{\mathfrak{G}}$ of the representation $U^L \times U^L$ of $\mathfrak{G} \times \mathfrak{G}$ and that by Lemma 2 of [4] $U^L \times U^L$ is equivalent to $U^{L \times L}$. Thus we may identify the space of $U^L \otimes U^L$ with the set of all functions $A, x, y \to A_{x,y}$ from $\mathfrak{G} \times \mathfrak{G}$ to the set of all linear

[1] A special case of this lemma is proved by essentially the same argument in [6].

operators from $\overline{\mathfrak{H}(L)}$ to $\mathfrak{H}(L)$ such that for all $x, y \in \mathfrak{G} \times \mathfrak{G}$ and all $\xi, \eta \in G \times G$

$$(\dagger) \qquad A_{\xi x, \eta y} = L_{\xi} A_{x,y} L^{*}_{\eta}.$$

Moreover it is not difficult to verify that the adjoint operation in the space of $U^L \otimes U^L$ goes over into the operation which takes A into A' where $A'_{x,y} = A^{*}_{y,x}$. Finally $((U^L \otimes U^L)_s(A))_{x,y} = A_{xs,ys}$ for all x, y, and s in $\mathfrak{G}$. For each $G \times G : \tilde{\mathfrak{G}}$ double coset d in $\mathfrak{G} \times \mathfrak{G}$ let $\mathfrak{H}_d$ be the set of all A in $\mathfrak{H}(U^{L \times L})$ which vanish outside of d. It is clear that $\mathfrak{H}(U^{L \times L})$ is a direct sum over the double cosets of the $\mathfrak{H}_d$ and that each $\mathfrak{H}_d$ is invariant under the transformations $(U^L \otimes U^L)_s$. This decomposition of $U^L \otimes U^L$ is that described in Theorem 2 of [4]. In order to deal with U^L Ⓢ U^L and U^L Ⓐ U^L we must alter the decomposition so that it is invariant under $A \to A'$ as well. We note that $(\mathfrak{H}_d)' = \mathfrak{H}_{d'}$ where d' is the double coset consisting of all x, y with $y, x \in d$. Let us call d' the transpose of d. Further let us denote $d \cup d'$ by $\bar{d}$ and let $\mathfrak{B}$ denote the set of all subsets of $\mathfrak{G} \times \mathfrak{G}$ of the form $\bar{d}$. If $b \in \mathfrak{B}$ let $\mathfrak{H}_b = \mathfrak{H}_d$ or $\mathfrak{H}_d \oplus \mathfrak{H}_{d'}$ according as $d = d'$ or not. Then $\mathfrak{H}(U^L \otimes U^L)$ is a direct sum of the $\mathfrak{H}_b$ for $b \in \mathfrak{B}$ and each $\mathfrak{H}_b$ is invariant under $A \to A'$ as well as under the operators $(U^L \otimes U^L)_s$. Let $\mathfrak{H}^{+}_b$ be the set of all $A \in \mathfrak{H}_b$ with $A = A'$ and let $\mathfrak{H}^{-}_b$ be the set of all $A \in \mathfrak{H}$ with $A = -A'$. Let V^{b+} and V^{b-} be the restrictions to $\mathfrak{H}^{+}_b$ and $\mathfrak{H}^{-}_b$ of $U^L \otimes U^L$. Then U^L Ⓢ $U^L = \sum_{b \in \mathfrak{B}} V^{b+}$ and U^L Ⓐ $U^L = \sum_{b \in \mathfrak{B}} V^{b-}$. These are the decompositions with which our theorem deals and all that remains is to identify the summands with specific induced representations of $\mathfrak{G}$. There are three cases according as b contains two distinct double cosets, coincides with $(G \times G)\mathfrak{G}$ or is a single double coset distinct from $(G \times G)\tilde{\mathfrak{G}}$. The first is readily disposed of. The second and third require a more elaborate discussion.

Case I. *b contains two distinct double cosets.* Then $\mathfrak{H}_b = \mathfrak{H}_d \oplus \mathfrak{H}_{d'}$. Clearly $\mathfrak{H}^{+}_b$ consists of all $A + A'$ where $A \in \mathfrak{H}_d$ and $\mathfrak{H}^{-}_b$ consists of all $A - A'$ where $A \in \mathfrak{H}_d$. Consider the linear mappings $A \to A + A'$ and $A \to A - A'$. These map $\mathfrak{H}_d$ in a one to one linear manner onto $\mathfrak{H}^{+}_b$ and $\mathfrak{H}^{-}_b$ respectively and since $A \to A'$ commutes with all $(U^L \otimes U^L)_s$ they set up equivalences between $U^L \otimes U^L$ restricted to $\mathfrak{H}_d$ and V^{b+} and V^{b-} respectively. Thus the four representations V^{b+}, V^{b-}, $U^L \otimes U^L$ restricted to $\mathfrak{H}_d$ and $U^L \otimes U^L$ restricted to $\mathfrak{H}_{d'}$ are all equivalent. But the latter two representations have already been identified in Theorem 2 of [4]. Indeed choose any x, y in b. Let xLx^{-1} and yLy^{-1} denote the representations $\xi \to L_{x\xi x^{-1}}$, $\eta \to L_{y\eta y^{-1}}$ of $x^{-1}Gx \cap y^{-1}Gy$. Let $M = (xLx^{-1}) \otimes (yLy^{-1})$. Then it follows

from the preceding discussion and Theorem 2 of [4] that V^{b+} and V^{b-} are both equivalent to the representation U^M of $\mathfrak{G}$ induced by the representation M of $x^{-1}Gx \cap y^{-1}Gy$.

Cases II *and* III. *b consists of a single double coset d with* $d = d'$. Let x, y be an arbitrary element of d. It follows from (†) that each A in $\mathfrak{H}_b = \mathfrak{H}_d$ is determined throughout b by its values on the left coset $(x, y)\tilde{\mathfrak{G}}$. Given A let us define a function B^A on $\mathfrak{G}$ as follows: $B^A{}_t = A_{xt,yt}$ for all t in $\mathfrak{G}$. Then (x, y being held fixed) B^A is determined by A and in turn determines A via the equation:

$$A_{\xi xt, \eta yt} = L_\xi B^A{}_t L^*{}_\eta \qquad (\dagger\dagger)$$

valid for all $\xi, \eta \in G \times G$ and all $t \in \mathfrak{G}$. A straightforward calculation now shows that an arbitrary function B from $\mathfrak{G}$ to $\mathfrak{H}(L \otimes L)$ is of the form B^A for some A in $\mathfrak{H}_b$ if and only if

$$B_{\zeta t} = L_{x\zeta x^{-1}}(B_t)\ L^*{}_{y\zeta y^{-1}} \qquad (\dagger\dagger\dagger)$$

for all $\zeta \in x^{-1}Gx \cap y^{-1}Gy$ and that moreover if $A_1 = (U^L \otimes U^L)_s A$ then $B^{A_1}{}_t = B^A{}_{ts}$. We now investigate the effect on B^A of replacing A by A'. That is given B we find A so that $B = B^A$ and then express $B^{A'}$ in terms of B. Since $A'_{x,y} = A^*_{y,x}$ we have $B'_t =$ (by definition) $B^{A'}{}_t = A'_{xt,yt} = A^*_{yt,xt}$. If we can find $\xi, \eta \in G \times G$ and a mapping s of $\mathfrak{G}$ into $\mathfrak{G}$ such that $\xi xs(t) = yt$ and $\eta ys(t) = xt$ for all $t \in \mathfrak{G}$ then

$$\begin{aligned} B^{A'}{}_t = A^*{}_{\xi xs(t), \eta ys(t)} &= (L_\xi(A_{xs(t),ys(t)})L^*{}_\eta)^* \\ &= (L_\xi B^A{}_{s(t)} L^*{}_\eta)^* = L_\eta (B^A{}_{s(t)})^* L^*{}_\xi. \end{aligned}$$

Thus $A \to A'$ will go over into $B \to B'$ where $B'_t = L_\eta B^*{}_{s(t)} L^*{}_\xi$. Now the equations $\xi xs(t) = yt$ and $\eta ys(t) = xt$ are equivalent to $s(t) = x^{-1}\xi^{-1}yt = y^{-1}\eta^{-1}xt$. Thus we must choose ξ and η so that $x^{-1}\xi^{-1}y = y^{-1}\eta^{-1}x$ and then set $s(t) = zt$ where $z = x^{-1}\xi^{-1}y = y^{-1}\eta^{-1}x$. But $x^{-1}\xi^{-1}y = y^{-1}\eta^{-1}x$ if and only if $\eta yx^{-1}\xi^{-1} = xy^{-1}$; that is if and only if xy^{-1} and its inverse are in the same $G : G$ double coset. Now as was pointed out in [4] the $G : G$ double cosets in $\mathfrak{G}$ and the $G \times G : \tilde{\mathfrak{G}}$ double cosets in $\mathfrak{G} \times \mathfrak{G}$ are in one-to-one correspondence in such a manner that $G \times G(x, y)\tilde{\mathfrak{G}}$ corresponds to $Gxy^{-1}G$ and it is clear that xy^{-1} lies in a self inverse $G : G$ double coset if and only if x, y and y, x lie in the same $G \times G : \tilde{\mathfrak{G}}$ double coset. In short ξ and η may be found if and only if b consists of a single double coset and this is the case under discussion. Thus $x^{-1}Gy \cap y^{-1}Gx$ is not empty and if we let z be any one of its elements we have $B'_t = L_\eta B^*{}_{zt} L^*{}_\xi$ where

$\eta = xz^{-1}y^{-1}$ and $\xi = yz^{-1}x^{-1}$. It is convenient to summarize the argument up to this point in a lemma.

LEMMA a. *If d is a $G \times G : \tilde{\mathfrak{G}}$ double coset such that $d = d'$ and x, y is any element of d then the representation of $\mathfrak{G}$ obtained by restricting $U^L \otimes U^L$ to $\mathfrak{H}_d$ is equivalent to the representation U^M where M is the representation $(xLx^{-1}) \otimes (yLy^{-1})$ of $x^{-1}Gx \cap y^{-1}Gy$. Moreover $x^{-1}Gy \cap y^{-1}Gx$ is not empty and if z is any one of its elements then the involution $B \to B'$ whose 1 and -1 spaces are the spaces of V^{b+} and V^{b-} respectively takes the following form:*

$$B'_t = L_{xz^{-1}y^{-1}}(B^*_{zt})L^*_{yz^{-1}x^{-1}}.$$

We also have

LEMMA b. *Let d, x, y and z be as in Lemma* a *and let $G_0 = x^{-1}Gx \cap y^{-1}Gy$. Then $z^2 \varepsilon G_0$ and $zG_0z^{-1} = G_0$ so that z and G_0 generate a subgroup G_1 of $\mathfrak{G}$ which contains G_0 as a normal subgroup of index two.*

Proof. Since

$$z \,\varepsilon\, x^{-1}Gy \cap y^{-1}Gx, \quad z^2 \,\varepsilon\, x^{-1}Gyy^{-1}Gx = x^{-1}Gx \text{ and } z^2 \,\varepsilon\, y^{-1}Gxx^{-1}Gy = y^{-1}Gy.$$

Therefore $z^2 \varepsilon G_0$. Moreover

$$z(x^{-1}Gx)z^{-1} = y^{-1}\xi xx^{-1}Gxx^{-1}\xi^{-1}y = y^{-1}Gy$$

and

$$z(y^{-1}Gy)z^{-1} = x^{-1}\eta yy^{-1}Gyy^{-1}\eta^{-1}x = x^{-1}Gx$$

where $\xi \varepsilon G$ and $\eta \varepsilon G$. Thus the inner automorphism defined by z interchanges the two groups of which G_0 is the intersection and hence leaves G_0 invariant.

LEMMA c. *Let the terminology be as in the two preceding lemmas. Then $G_0 = G_1$ if and only if $d = (G \times G)\tilde{\mathfrak{G}}$.*

Proof. Let $d = (G \times G)\tilde{\mathfrak{G}}$. Then $x = \xi a$ and $y = \eta a$ where $a \varepsilon \mathfrak{G}$ and $\xi, \eta \varepsilon G \times G$. Hence

$$x^{-1}Gy \cap y^{-1}Gx = a^{-1}Ga \cap a^{-1}Ga = a^{-1}Ga$$

and $G_0 = a^{-1}Ga \cap a^{-1}Ga = a^{-1}Ga$. It follows that $z \varepsilon G_0$ and $G_0 = G_1$. Conversely if $G_0 = G_1$ so that $z \varepsilon G_0$ then there exists $\xi \varepsilon G$ so that $x^{-1}\xi y \varepsilon x^{-1}Gx \cap y^{-1}Gy$. Hence $y \varepsilon Gx$. Hence $x, y \varepsilon (G \times G)\tilde{\mathfrak{G}}$. Therefore $d = (G \times G)\tilde{\mathfrak{G}}$.

We now discuss cases II and III separately.

Case II. $d = (G \times G)\tilde{\mathfrak{G}}$. Here we may choose $x = y = z = e$ where e

is the identity of $\mathfrak{G}$ and the expression for B'_t in Lemma a simplifies to $B'_t = B^*_t$. Thus $B = B'$ if and only if $B_t = B^*_t$ for all t and $B' = -B$ if and only if $B_t = -B^*_t$ for all t. It should now be evident that in this case V^{b+} and V^{b-} are equivalent to $U^{L \circledS L}$ and $U^{L \circledA L}$ respectively.

Case III. $d = d'$ but $d \neq (G \times G)\tilde{\mathfrak{G}}$. In this case $z \notin G_0$ and G_0 is a proper subgroup of G_1. We know that $V^{b+} + V^{b-} = U^M$ where M is the representation $(xLx^{-1}) \otimes (yLy^{-1})$ of G_0. We shall identify V^{b+} and V^{b-} by proving them equivalent to the induced representations U^{M^+} and U^{M^-} where M^+ and M^- are certain extensions of M to the group G_1. To this end let T be the non singular linear operator in $\mathfrak{H}(L \otimes L) = \mathfrak{H}(M)$ which takes S into $L_{xz^{-1}y^{-1}}(S^*) L^*_{yz^{-1}x^{-1}}$. Then

$$T^2(S) = L_{xz^{-1}y^{-1}}L_{yz^{-1}x^{-1}}(S)L^*_{xz^{-1}y^{-1}}L^*_{yz^{-1}x^{-1}} = L_{xz^{-2}x^{-1}}(S)L_{yz^{-2}y^{-1}} = M_{z^{-2}}(S).$$

Thus $T^2 = M_{z^{-2}}$ and this suggests that we might be able to extend M to G_1 by defining M_z to be T^{-1} or $-T^{-1}$. Of course an extension of M to G_1 is uniquely determined by its value Q at z and if such an extension exists we surely have $Q^2 = M_{z^2}$ and $M_{z\zeta z^{-1}}QM_\zeta Q^{-1}$ for all $\zeta \in G_0$. Conversely it is easy to see that given any Q satisfying these two conditions there exists a unique extension of M to G_1 such that $M_z = Q$. Thus if we can verify that $M_{z\zeta z^{-1}} = T^{-1}M_\zeta T$ for all $\zeta \in G_0$ we will be assured that there exist unique extensions M^+ and M^- of M to G_1 such that $M^+_z = T^{-1}$ and $M^-_z = -T^{-1}$. But

$$\begin{aligned}(T^{-1}M_\zeta T)(S) &= T^{-1}M_\zeta(L_{xz^{-1}y^{-1}}(S^*)L^*_{yz^{-1}x^{-1}}) = T^{-1}(L_{x\zeta x^{-1}}L_{xz^{-1}y^{-1}}(S^*) \\ &\quad L^*_{yz^{-1}x^{-1}}L^*_{y\zeta y^{-1}}) = T^{-1}(L_{x\zeta z^{-1}y^{-1}}S^*L^*_{yz^{-1}x^{-1}}) = (L_{yzx^{-1}}L_{x\zeta z^{-1}y^{-1}}(S^*) \\ &\quad L^*_{y\zeta z^{-1}x^{-1}}L^*_{xyz^{-1}})^* = L_{xz\zeta z^{-1}x^{-1}}(S)L^*_{yz\zeta z^{-1}y^{-1}} = M_{z\zeta z^{-1}}(S).\end{aligned}$$

Thus T has the required properties and may be extended to M^+ and M^- as indicated. Finally note that $B'_t = T(B_{zt})$ for all t. Thus $B'_t \equiv B_t$ if and only if $B_{zt} \equiv T^{-1}(B_t)$; that is $B_{zt} \equiv M^+_z(B_t)$. Similarly $B' \equiv -B_t$ if and only if $B_{zt} \equiv M^-_z(B_t)$. Since we know already that $B_{\zeta t} = M_\zeta(B_t)$ for all $t \in \mathfrak{G}$ and all $\zeta \in G_0$ we see finally that $\mathfrak{H}(V^{b+})$ is the set of all functions B from $\mathfrak{G}$ to $\mathfrak{H}(M^+)$ such that $B_{\theta t} = M^+_\theta(B_t)$ for all $\theta \in G_1$ and all $t \in \mathfrak{G}$ and that $\mathfrak{H}(V^{b-})$ is similarly related to M^-. In other words we see that V^{b+} and V^{b-} are equivalent respectively to the induced representations U^{M^+} and U^{M^-}. We have now completed the proof of our main theorem which we may state as follows.

THEOREM 1. *Let L be a representation of the subgroup G of the finite group $\mathfrak{G}$ and let the field of the vector space $\mathfrak{H}(L)$ be of characteristic*

different from two. Let $\mathfrak{B}_1$ *be the set of all self inverse* $G:G$ *double cosets in* $\mathfrak{G}$ *except* G *itself. Let* $\mathfrak{B}_2$ *be the set of all sets of the form* $d \cup d'$ *where* d *is a non self inverse* $G:G$ *double coset and* d' *is the inverse of* d. *For each* $b \varepsilon \mathfrak{B}_2$ *choose* x *and* y *so that* $xy^{-1} \varepsilon b$ *and let* M *be the representation* $xLx^{-1} \otimes yLy^{-1}$ *of* $G_0 = x^{-1}Gx \cap y^{-1}Gy$. *Then the induced representation* U^M *of* $\mathfrak{G}$ *is independent of the choice of* x *and* y *and may be denoted by* V^b. *For each* $b \varepsilon \mathfrak{B}_1$ *choose* x *and* y *so that* $xy^{-1} \varepsilon b$. *Then* $x^{-1}Gy \cap y^{-1}Gx$ *is not empty. Let* z *be any one of its members. Let* G_0 *be as defined above and let* G_1 *be the subgroup generated by* G_0 *and* z. *Then* G_0 *is a normal subgroup of* G_1 *of index two. Let* M *be the representation of* G_0 *defined as described above and let* T *be the linear transformation in* $\mathfrak{H}(M)$ *which takes* S *into* $L_{xz^{-1}y^{-1}}(S^*)L^*_{yz^{-1}x^{-1}}$. *Then there exists unique extensions* M^+ *and* M^- *of* M *to* G_1 *such that* $M^+_z = T^{-1}$ *and* $M^-_z = -T^{-1}$. *The induced representations* U^{M^+} *and* U^{M^-} *are independent of* x, y *and* z *and may be denoted by* V^{b+} *and* V^{b-} *respectively. Finally we have*

$$U^L \circledS U^L = U^{L \circledS L} + \sum_{b \varepsilon \mathfrak{B}_1} V^{b+} + \sum_{b \varepsilon \mathfrak{B}_2} V^b,$$

$$U^L \circledA U^L = U^{L \circledA L} + \sum_{b \varepsilon \mathfrak{B}_1} V^{b-} + \sum_{b \varepsilon \mathfrak{B}_2} V^b.$$

COROLLARY 1. *The symmetric (resp. anti symmetric) Kronecker square of an induced representation is equivalent to a direct sum of induced representations.*

When L is one dimensional (that is, when U^L is monomial) so that L may be regarded as an $\mathfrak{F}$ valued function the descriptions of M^+ and M^- may be somewhat simplified. M is then also an $\mathfrak{F}$ valued function and it may be extended to G_1 so as to be a homomorphism of G_1 into $\mathfrak{F}$ in exactly two ways; namely by defining its value at z to be $L(xz^2x^{-1})$ or to be $-L(xz^2x^{-1})$. These two extensions are the M^+ and M^- of the theorem. We note correspondingly the corollary.

COROLLARY 2. *The symmetric (respectively anti symmetric) Kronecker square of a monomial representation is equivalent to a direct sum of monomial representations.*

Because of the relationship between intertwining operators and Kronecker products it is easy to deduce from Theorem 1 formulae for computing $\mathfrak{I}_S(\overline{U^L}, U^L)$ and $\mathfrak{I}_A(\overline{U^L}, U^L)$. The deduction is straight-forward and we shall content ourselves with a statement of the result. Moreover in order to avoid excessively complicated statements we shall confine ourselves to the case in which L is one dimensional and hence a character of G.

THEOREM 2. *Let L, G, $\mathfrak{G}$, $\mathfrak{F}$, $\mathfrak{B}_1$, $\mathfrak{B}_2$, be as in Theorem 1 except for the additional assumption that L is one dimensional. For each $b \varepsilon \mathfrak{B}_2$ choose x and y so that $xy^{-1} \varepsilon b$ and let M be the character $(xLx^{-1})(yLy^{-1})$. Then whether or not $M(\zeta) \equiv 1$ depends only upon b and we may write $j(b) = 1$ if $M(\zeta) \equiv 1$ and $j(b) = 0$ if $M(\zeta) \not\equiv 1$. For each $b \varepsilon \mathfrak{B}_1$ choose x and y so that $xy^{-1} \varepsilon b$. Then there exists $z \varepsilon x^{-1}Gy \cap y^{-1}Gx$ and if G_1 is the subgroup generated by $G_0 = x^{-1}Gx \cap y^{-1}Gy$ and z then G_0 is a normal subgroup of G_1 of index two. If M is defined as above then M may be extended so as to be a character of G_1 in exactly two ways as follows: $M^+(z) = L(xz^2x^{-1})$, $M^-(z) = -L(xz^2x^{-1})$. Whether or not $M(\zeta) \equiv 1$ depends only upon b. Moreover if $M(\zeta) \equiv 1$ so that $L(xz^2x^{-1}) = \pm 1$ then whether the plus or minus sign occurs depends only upon b. If $M(\zeta) \not\equiv 1$ we set $j(b) = 0$. If $M(\zeta) \equiv 1$ we set $j(b) = L(xz^2x^{-1})$. Then*

$$\mathfrak{I}_S(\overline{U^L}, U^L) = \mathfrak{I}_S(\bar{L}, L) + \sum_{b \varepsilon \mathfrak{B}_1} \tfrac{1}{2}(j(b)(1 + j(b))) + \sum_{b \varepsilon \mathfrak{B}_2} j(b),$$

$$\mathfrak{I}_A(\overline{U^L}, U^L) = \mathfrak{I}_A(\bar{L}, L) + \sum_{b \varepsilon \mathfrak{B}_1} \tfrac{1}{2}(j(b)(-1 + j(b))) + \sum_{b \varepsilon \mathfrak{B}_2} j(b).$$

Subtracting the two equations of the conclusion of the theorem we find the corollary:

COROLLARY 1. $c(U^L) = c(L) + \sum_{b \varepsilon \mathfrak{B}_1} j(b)$.

If in particular L is the identity so that U^L is a permutation representation then $j(b) = 1$ for all $b \varepsilon \mathfrak{B}_1$ and we have

COROLLARY 2. *If L is the identity representation of the subgroup G of $\mathfrak{G}$ then $c(U^L)$ is equal to the number of self inverse $G:G$ double cosets in $\mathfrak{G}$.*

In the special case in which U^L is completely reducible we know from the second corollary of Lemma 1 that $c(U^L)$ is the sum of the multiplicities of those irreducible components V of U^L for which $c(V) = 1$ minus the sum of the multiplicities of those irreducible components V of U^L for which $c(V) = -1$. Thus Corollary 2 includes the first main theorem of Frame's paper [1]. It should be noted that the special case of Theorem 2 in which L is the identity representation can be proved directly quite easily without appeal to the somewhat complicated general Theorem 1. Since the applications that we give in the present paper involve only this special case it seems worthwhile to give this proof explicitly.

THEOREM 2′. *Let G be a subgroup of the finite group $\mathfrak{G}$. Let I be the identity representation of G with respect to a field $\mathfrak{F}$ whose characteristic is not equal to two. Let n_1 denote the number of self inverse $G:G$ double cosets*

in $\mathfrak{G}$ and let n_2 denote the number of non self inverse $G:G$ double cosets in $\mathfrak{G}$. Then $\mathfrak{I}_S(U^I, U^I) = n_1 + \frac{1}{2}(n_2)$ and $\mathfrak{I}_A(U^I, U^I) = \frac{1}{2}(n_2)$.

Proof. (direct) By the argument at the beginning of section two, the general linear transformation from $\mathfrak{H}(\overline{U^I})$ into $\mathfrak{H}(U^I)$ will be defined by an $\mathfrak{F}$ valued function A on $\mathfrak{G} \times \mathfrak{G}$ such that $A_{\xi x, \eta y} = A_{x,y}$ for all $\xi, \eta \varepsilon G \times G$ and all $x, y \varepsilon \mathfrak{G} \times \mathfrak{G}$. This operator will be an intertwining operator if and only if $A_{xs,ys} = A_{x,y}$ for all x, y, and s in $\mathfrak{G}$. In short the intertwining operators correspond in a one-to-one linear fashion to the $\mathfrak{F}$ valued functions on $\mathfrak{G} \times \mathfrak{G}$ which are constant on the $G \times G : \tilde{\mathfrak{G}}$ double cosets—or simply to the $\mathfrak{F}$ valued functions on the $(G \times G) : \tilde{\mathfrak{G}}$ double cosets. In this correspondence the adjoint of the operation defined by $d \rightarrow A(d)$ is simply $d \rightarrow A(d')$ where d' is the set of all x, y with $y, x \varepsilon d$. Since $A(d) = -A(d')$ is impossible when $d = d'$ unless $A(d') = 0$ the theorem is now an obvious consequence of the correspondence between $G:G$ double cosets in $\mathfrak{G}$ and $G \times G : \tilde{\mathfrak{G}}$ double cosets in $\mathfrak{G} \times \mathfrak{G}$.

3. Simply reducible groups. Wigner in [6] has defined a finite group $\mathfrak{G}$ to be *simply reducible* if it has the following two properties: (a) Every representation L is equivalent to its adjoint $\bar{L}$. (b) Every Kronecker product $L \otimes M$ of irreducible representations is a direct sum of irreducible representations each of which occurs with multiplicity one. The main result of [6] is the following curious characterization of simply reducible groups. $\mathfrak{G}$ is simply reducible if and only if $\sum_{x \varepsilon G} \zeta(x)^3 = \sum_{x \varepsilon G} v(x)^2$ where for each x in $\mathfrak{G}$, $\zeta(x)$ is the number of square roots of x and $v(x)$ is the number of elements which commute with x. It is also shown in [6] that for arbitrary groups the left hand side of the above equation is less than or equal to the right hand side. In this section we shall establish a somewhat different characterization of simply reducible groups which is like Wigner's in that it is expressed in terms independent of representation theory but has the advantage that it is significant for infinite groups as well. In the next section we shall discuss the connection between the two characterizations and show how to derive one from the other. The chief tools in our approach are Theorem 2′ and the following easy consequence of Lemma 1. We shall assume from now on that $\mathfrak{F}$ is algebraically closed and of characteristic which is not equal to two and does not divide the order of $\mathfrak{G}$.

LEMMA 5. *Let $M = M_1 + M_2 + \cdots + M_n$ where the M_j are irreducible representations of the finite group $\mathfrak{G}$. Then $\mathfrak{I}_A(\bar{M}, M) = 0$ if and only if for each j either*

(a) $c(M_j) = 1$ *and* M_j *is not equivalent to* M_k *for any* $k \neq j$

or

(b) $c(M_j) = 0$ *and* $\bar{M}_j$ *is not a component of* M.

Proof. For each j let $M^0{}_j = M_1 + M_2 + \cdots M_{j-1} + M_{j+1} + \cdots + M_n$. By Lemma 1, $\mathfrak{I}_A(\bar{M}, M) = \mathfrak{I}_A(\bar{M}_j, M_j) + \mathfrak{I}_A(\bar{M}^0{}_j, M^0{}_j) + \mathfrak{I}(\bar{M}_j, M^0{}_j)$. Hence $\mathfrak{I}_A(\bar{M}, M) = 0$ if and only if all j we have $\mathfrak{I}_A(\bar{M}_j, M_j) = 0$ and for all k with M_k a component of $M^0{}_j$ we have $\mathfrak{I}(\bar{M}_j, M_k) = 0$. Now the first part of this last statement is equivalent to the statement that $c(M_j) = 1$ or 0. Moreover if $c(M_j) = 1$ then $M_j = \bar{M}_j$ so the second part is equivalent to the statement that M_j does not occur in $M^0{}_j$; that is that M_j occurs with multiplicity one. If $c(M_j) = 0$ the second part is equivalent to the statement that $\bar{M}_j$ does not occur at all. Thus the lemma is proved.

If $M = \bar{M}$ then alternative (b) is impossible and we have the

COROLLARY. *If* $M = \bar{M}$ *then* $\mathfrak{I}_A(\bar{M}, M) = 0$ *if and only if* M *is a direct sum of distinct irreducible components* M_j *with* $c(M_j) = 1$.

THEOREM 3. *If* I *is the identity representation of the subgroup* G *of the finite group* $\mathfrak{G}$ *then the following statements are equivalent:*

(a) $\mathfrak{I}_A(\overline{U^I}, U^I) = 0$.

(b) *Every* $G:G$ *double coset is self inverse.*

(c) *The irreducible components* M_j *of* U^I *occur with multiplicity one and for all* $j, c(M_j) = 1$.

Proof. The equivalence of (a) and (c) follows from the corollary to Lemma 5. The equivalence of (a) and (b) follows from Theorem 2′.

We obtain our characterization of simply reducible groups by applying Theorem 3 with a suitably chosen G and $\mathfrak{G}$. Let $\mathfrak{G}$ be an arbitrary finite group and let $\tilde{\mathfrak{G}}_3$ denote the diagonal subgroup of the triple Cartesian product $\mathfrak{G}_3 = \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}$ of $\mathfrak{G}$ with itself; that is let $\tilde{\mathfrak{G}}_3$ be the set of all x, y, z with $x = y = z$. Let I be the identity representation of $\tilde{\mathfrak{G}}_3$ and let U^I be the representation of $\mathfrak{G}_3$ induced by I. The general irreducible representation of $\mathfrak{G}_3$ is $L \times M \times N$ where L, M, and N are arbitrary irreducible representations of $\mathfrak{G}$. Moreover by the Frobenius reciprocity theorem (which may be obtained as a corollary of Theorem 2 of [4]) the multiplicity of occurrence of $L \times M \times N$ in U^I is equal to the multiplicity of occurrence of I in $L \otimes M \otimes N$; that is to the multiplicity of occurrence of $\bar{L}$ in $M \otimes N$.

Thus a necessary and sufficient condition that all irreducible components of U^1 occur with multiplicity one is that for all irreducible representations M and N of $\mathfrak{G}$ all irreducible components of $M \otimes N$ occur with multiplicity one. In short $\mathfrak{G}$ satisfies (b) of the definition of simple reducibility if and only if U^1 satisfies the first part of (c) under Theorem 3. On the other hand since $c(L \times M \times N) = c(L)c(M)c(N)$ a necessary and sufficient condition that the second part of (c) should be satisfied is that $c(L)c(M)c(N) = 1$ whenever L appears as an irreducible component of $M \otimes N$. But applying Lemma 4 to the restriction of $M \times N$ to the diagonal in $\mathfrak{G} \times \mathfrak{G}$ we see that this is the case whenever $M \otimes N$ is free of multiplicities and $c(M)c(N) \neq 0$; that is for all M and N if $\mathfrak{G}$ is simply reducible. Note next that if $c(L)c(M)c(N) = 1$ whenever $\bar{L}$ appears in $M \otimes N$ then $c(M) \neq 0$ for any M. Thus the second part of (c) under Theorem 3 implies (a) in the definition of simple reducibility. We have now proved that $\mathfrak{G}$ is simply reducible if and only if U^1 satisfies (c) under Theorem 3. Applying Theorem 3 we conclude at once the truth of

THEOREM 4. *Let $\mathfrak{G}$ be a finite group and let $\tilde{\mathfrak{G}}_3$ be the diagonal subgroup in $\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}$. Then $\mathfrak{G}$ is simply reducible if and only if every $\tilde{\mathfrak{G}}_3 : \tilde{\mathfrak{G}}_3$ double coset in $\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}$ is self inverse.*

4. Wigner's condition. We shall relate our characterization of simply reducible groups to Wigner's by interpreting the two sides of his equation. This will lead us to a generalization of Wigner's inequality and to further theorems like Theorem 4.

LEMMA 6. *Let $\mathfrak{G}$ act as a group of permutations on a finite set S. Let $\mathfrak{G}$ contain h elements and let T be a permutation of S which commutes with the permutations defined by the members of $\mathfrak{G}$. For each $y \varepsilon \mathfrak{G}$ let $p(y)$ be the number of elements s of S such that $y(s) = T(s)$. Then $(1/h) \sum_{y \varepsilon G} p(y)$ is equal to the number of orbits of S under $\mathfrak{G}$ which are invariant under T.*

Proof. For each $s, y \varepsilon S \times \mathfrak{G}$ let $k(s,y) = 0$ or 1 according as $y(s) \neq T(s)$ or $y(s) = T(s)$. For each $s \varepsilon S$ let $q(s)$ be the number of elements y in $\mathfrak{G}$ such that $y(s) = T(s)$. Then

$$\sum_{y \varepsilon G} p(y) = \sum_{y \varepsilon G} \sum_{s \varepsilon S} k(s,y) = \sum_{s \varepsilon S} \sum_{y \varepsilon G} k(s,y) = \sum_{s \varepsilon S} q(s).$$

But $q(s)$ is zero if $T(s)$ is in the orbit containing s and h/n_s where n_s is the number of elements contained in orbit containing s if $T(s)$ is not in the orbit

containing s. Hence $\sum_{y \varepsilon G} p(y) = \sum'_{s \varepsilon S} (h/n_s)$ where the $'$ indicates that all s in non T invariant orbits have been omitted. Now for each T invariant orbit the term h/n_s occurs n_s times. Hence $\sum_{y \varepsilon G} p(y)$ is equal to h times the number of T invariant orbits as was to be proved.

THEOREM 5. *Let $\mathfrak{G}$ be a finite group of order h and let $\mathfrak{G}_{n+1}$ be the direct product of $\mathfrak{G}$ with itself $n+1$ times where $n \geqq 1$. Let $\tilde{\mathfrak{G}}_{n+1}$ be the diagonal subgroup of $\mathfrak{G}_{n+1}$. For each $x \varepsilon \mathfrak{G}$ let $v(x)$ be the number of elements of $\mathfrak{G}$ which commute with x. Then the following three numbers are equal:*

(a) $(1/h) \sum_{x \varepsilon G} v(x)^n$.

(b) *The number of $\tilde{\mathfrak{G}}_{n+1} : \tilde{\mathfrak{G}}_{n+1}$ double cosets in $\mathfrak{G}_{n+1}$.*

(c) *The number of orbits in $\mathfrak{G}_n$ under the group of inner automorphisms defined by members of the diagonal of $\mathfrak{G}_n$.*

Proof. The equality of (b) and (c) is an obvious consequence of their definitions and that of (a) and (c) follows at once from Lemma 6. We need only let $S = \mathfrak{G}_n$, $y(x_1, x_2, \cdots, x_n) = y^{-1}x_1y, y^{-1}x_2y, \cdots, y^{-1}x_ny$ and $T(x_1, x_2, \cdots, x_n) = x_1, x_2, \cdots, x_n$. $p(y)$ is clearly $v(y)^n$.

THEOREM 6. *Let $\mathfrak{G}$, $\mathfrak{G}_{n+1}$ and $\tilde{\mathfrak{G}}_{n+1}$ be as in Theorem 5. For each $x \varepsilon \mathfrak{G}$ let $\zeta(x)$ be the number of square roots of x. Then the following three numbers are all equal:*

(a) $(1/h) \sum_{x \varepsilon G} \zeta(x)^{n+1}$.

(b) *The number of self inverse $\tilde{\mathfrak{G}}_{n+1} : \tilde{\mathfrak{G}}_{n+1}$ double cosets in $\mathfrak{G}_{n+1}$.*

(c) *The number of self inverse orbits in $\mathfrak{G}_n$ under the group of inner automorphisms defined by members of the diagonal of $\mathfrak{G}_n$.*

Proof. The equality of (b) and (c) is an obvious consequence of the definitions. Applying Lemma 6 as in the proof of Theorem 5 but now taking $T(x_1, x_2, \cdots, x_n) = x_1^{-1}, x_2^{-1}, \cdots, x_n^{-1}$ we find that (c) is equal to $\sum_{y \varepsilon G} \zeta_1(y)^n$ where for each y in $\mathfrak{G}$, $\zeta_1(y)$ is the number of x in $\mathfrak{G}$ such that $y^{-1}xy = x^{-1}$. But $y^{-1}xy = x^{-1}$ if and only if $xyx = y$; that is if and only if $(xy)^2 = y^2$. Hence $\zeta_1(y) = \zeta(y^2)$. Hence $\sum_{y \varepsilon G} \zeta_1(y)^n = \sum_{y \varepsilon G} \zeta(y^2)^n = \sum_{z \varepsilon G} (\zeta(z))^n$ (number of y with $y^2 = z$) $= \sum_{z \varepsilon G} (\zeta(z))^{n+1}$.

As an immediate consequence of Theorems 4 and 5 we derive

THEOREM 7. *Let $\mathfrak{G}$ be any finite group and let v, ζ, $\mathfrak{G}_n$, $\tilde{\mathfrak{G}}_n$ be defined as in Theorems 5 and 6. Then for all $n = 1, 2, \cdots$ we have*

$$\sum_{x \in G} \zeta(x)^{n+1} \leqq \sum_{x \in G} v(x)^n.$$

Equality holds if and only if every $\tilde{\mathfrak{G}}_{n+1} : \tilde{\mathfrak{G}}_{n+1}$ double coset in $\mathfrak{G}_{n+1}$ is self inverse.

Specializing to $n = 2$ and applying Theorem 4 we obtain at once the inequality and the characterization of simply reducible groups which is Theorem 2 of Wigner's paper [6]. If in Theorem 6 we set $n = 1$ we find that $\sum_{x \in G} \zeta(x)^2$ is the order of $\mathfrak{G}$ multiplied by the number of self inverse classes in $\mathfrak{G}$. This is Wigner's Theorem 1.

It is natural to ask what equality means in terms of representation theory for values of n other than 2 and it turns out to be possible to give a complete answer.

THEOREM 8. *The following conditions on a finite group $\mathfrak{G}$ are equivalent*

(a) $\sum_{x \in G} \zeta(x)^2 = \sum_{x \in G} v(x)$.

(b) *Every class in $\mathfrak{G}$ is self inverse.*

(c) *For every representation L of $\mathfrak{G}$, L and $\bar{L}$ are equivalent.*

Proof. Let $\tilde{\mathfrak{G}}$ be the diagonal of $\mathfrak{G} \times \mathfrak{G}$. Let I be the identity representation of $\tilde{\mathfrak{G}}$. Then if L and M are irreducible representations of $\mathfrak{G}$, U^I contains $L \times M$ as a component just as many times as $L \otimes M$ contains I. But $L \times M$ contains I exactly once or not at all depending upon whether or not $M = \bar{L}$. Hence $U^I = \sum L \times \bar{L}$ where L ranges over all irreducible representations of $\mathfrak{G}$. Hence by Theorem 3 $\mathfrak{d}_A(\overline{U^I}, U^I) = 0$ if and only if $c(L \times \bar{L}) = 1$ for all L. But $c(L \times \bar{L}) = c(L)^2$. Thus $\mathfrak{d}_A(\overline{U^I}, U^I) = 0$ if and only if $L = \bar{L}$ for all L. It now follows from Theorem 3 that (b) and (c) are equivalent. That (a) and (b) are equivalent follows at once from Theorem 7.

It is to be remarked that the equivalence of (b) and (c) is a well known result.

THEOREM 9. *The following conditions on a finite group $\mathfrak{G}$ are equivalent.*

(a) *For some integer $n \geqq 3$, $\sum_{x \in G} \zeta(x)^{n+1} = \sum_{x \in G} v(x)^n$.*

(b) *For every positive integer n,* $\sum_{x \varepsilon G} \zeta(x)^{n+1} = \sum_{x \varepsilon G} v(x)^n$.

(c) *If L is any irreducible representation of $\mathfrak{G}$ then $L \otimes \bar{L}$ is the identity.*

(d) *$\mathfrak{G}$ is a direct product of groups of order two.*

Proof. It is obvious that (c) and (d) are equivalent and that (d) implies (a) and (b). It is also obvious that (b) implies (a). Now the condition: Every $\tilde{\mathfrak{G}}_{n+1}:\tilde{\mathfrak{G}}_{n+1}$ double coset in $\mathfrak{G}_{n+1}$ is self inverse can be reformulated to read: Given any n elements $x_1, x_2, \cdots, x_n$ in $\mathfrak{G}$ there exists $s \varepsilon \mathfrak{G}$ such that $sx_js^{-1} = x_j^{-1}$ for $j = 1, 2, \cdots, n$. Thus when the condition holds for any n it holds for all smaller n. Thus we need only prove that if (a) holds for $n = 3$ then (d) holds. But if (a) holds with $n = 3$ then by Theorem 7 and Theorem 3 we have $\mathfrak{I}_A(U^I, U^I) = 0$ where U^I is the representation of $\mathfrak{G} \times \mathfrak{G} \times \mathfrak{G} \times \mathfrak{G}$ induced by the identity representation of the diagonal $\tilde{\mathfrak{G}}_4$. Hence if L, M, N and K are arbitrary irreducible representations of $\mathfrak{G}$ then $L \times M \times N \times K$ occurs with multiplicity one or zero in U^I. Hence in particular $(L \otimes M) \otimes (\bar{L} \otimes \bar{M})$ contains the identity at most once. Hence $L \otimes M$ is irreducible for all irreducible L and M of $\mathfrak{G}$. Hence $L \otimes \bar{L}$ is irreducible for all L. But $L \otimes \bar{L}$ always contains the identity. Hence $L \otimes \bar{L}$ is the identity representation. Hence all irreducible representations of $\mathfrak{G}$ are one dimensional. Hence $\mathfrak{G}$ is Abelian and every element of its character group is of order 2. Since $\mathfrak{G}$ is isomorphic to its character group (d) follows and the theorem is proved.

5. Even and odd representations. Wigner calls an irreducible representation *even* if it occurs as a component in the reduction of $L \circledS L$ for some irreducible L with $c(L) = 1$ or in $L \circledA L$ for some irreducible L with $c(L) = -1$. He calls it *odd* if it occurs as a component in the reduction of $L \circledA L$ for some irreducible L with $c(L) = 1$ or[2] in $L \circledS L$ for some irreducible L with $c(L) = -1$. The last theorem of Wigner's paper (Theorem 3 of [6]) asserts that no representation of a simply reducible group can be both even and odd. In the language of intertwining numbers this theorem says that when $\mathfrak{G}$ is simply reducible and L and M are irreducible then $c(L)c(M) = 1$ implies that $\mathfrak{I}(L \circledS L, M \circledA M) = 0$ and $c(L)c(M) = -1$ implies that $\mathfrak{I}(L \circledS L, M \circledS M) = \mathfrak{I}(L \circledA L, M \circledA M) = 0$. On the other

[2] Actually this last phrase is omitted in Wigner's definition. For reasons of symmetry we believe that this omission must have been accidental. The theorem in question is stronger and still true when the definition is given as above.

hand when $\mathfrak{G}$ is simply reducible $L \otimes M$ is free of multiplicities and by Lemma 4 $c(V) = c(L)c(M)$ for all irreducible components V of $L \otimes M$. Thus if $c(L)c(M) = 1$ then $\mathfrak{I}_A(\overline{L \otimes M}, L \otimes M) = 0$ by the corollary to Lemam 5. By an obvious analogous argument if $c(L)c(M) = -1$ then $\mathfrak{I}_S(\overline{L \otimes M}, L \otimes M) = 0$. Thus Wigner's Theorem 3 is a consequence of the following lemma whose proof is an immediate consequence of the definitions.

LEMMA 7. *Let L and M be irreducible representations of an arbitrary finite group $\mathfrak{G}$. Then*

$$(L \circledS L) \otimes (M \circledA M) \subseteq (L \otimes M) \circledA (L \otimes M)$$
$$(L \circledS L) \otimes (M \circledS M) \subseteq (L \otimes M) \circledS (L \otimes M)$$
$$(L \circledA L) \otimes (M \circledA M) \subseteq (L \otimes M) \circledS (L \otimes M)$$

6. The conjugating representation of a group. Let $\mathfrak{H}$ be the set of all $\mathfrak{F}$ valued functions on the finite group $\mathfrak{G}$. For each $s \varepsilon \mathfrak{G}$ let A_s be the operator in $\mathfrak{H}$ such that $(A_s(f))(x) = f(s^{-1}xs)$. Then A is a representation of $\mathfrak{G}$ which Frame [2] has called the conjugating representation. We shall show that most of the results of Frame's paper follow at once from Theorem 1 of [4] and the first part of the proof of Theorem 8 of the present paper.

THEOREM 10. *Let $\mathfrak{G}$ be a finite group and let $\tilde{\mathfrak{G}}$ be the isomorphic replica furnished by the diagonal in $\mathfrak{G} \times \mathfrak{G}$. Let I be the identity representation of $\tilde{\mathfrak{G}}$. Then the following representations of $\mathfrak{G}$ are equivalent.*

(a) *The conjugating representation of $\tilde{\mathfrak{G}}$.*

(b) *The restriction to $\tilde{\mathfrak{G}}$ of the induced representation U^I of $\mathfrak{G} \times \mathfrak{G}$.*

(c) *$\sum_{c \varepsilon \mathfrak{C}} P_c$ where $\mathfrak{C}$ is the set of all classes in $\tilde{\mathfrak{G}}$ and P_c is the permutation representation of $\tilde{\mathfrak{G}}$ associated with the normalizer of any element in that class.*

(d) *$\sum L \otimes \bar{L}$ where L varies over all irreducible representations of $\tilde{\mathfrak{G}}$.*

Proof. $\mathfrak{H}(U^I)$ is the set of all $\mathfrak{F}$ valued functions f on $\mathfrak{G} \times \mathfrak{G}$ such that $f(tx, ty) = f(x, y)$ for all $t, x, y \varepsilon \mathfrak{G}$ and $(U^I{}_s(f))(x, y) = f(xs, ys)$. Let $(T(f))(z, z) = f(e, z)$ where e is the identity of $\mathfrak{G}$. Then T is one-to-one and linear from $\mathfrak{H}(U^I)$ to the set of all $\mathfrak{F}$ valued functions on $\tilde{\mathfrak{G}}$ and an obvious calculation shows that $s \rightarrow TU^I{}_sT^{-1}$ is the conjugating representation of $\tilde{\mathfrak{G}}$. Thus (a) and (b) are equivalent. Now by Theorem 1 of [4] U^I

restricted to $\tilde{\mathfrak{G}}$ is a sum over the double cosets of certain induced representations of $\tilde{\mathfrak{G}}$; the induced representation associated with the double coset containing x, y being that induced by the identity representation of $\tilde{\mathfrak{G}} \cap (x, y)^{-1}\tilde{\mathfrak{G}}(x, y)$. But this last subgroup is simply the set of all z, z such that $x^{-1}zx = y^{-1}zy$ or $(yx^{-1})z = z(yx^{-1})$; that is the normalizer of yx^{-1}. Since the classes in $\mathfrak{G}$ and the $\tilde{\mathfrak{G}} : \tilde{\mathfrak{G}}$ double cosets in $\mathfrak{G} \times \mathfrak{G}$ are in one-to-one correspondence in such a manner that x_1, y_1 and x_2, y_2 are in the same double coset if and only if $y_1 x^{-1}_1$ and $y_2 x^{-1}_2$ are in the same class, the equivalence of (b) and (c) follows at once. The first four sentences in the proof of Theorem 8 prove that $U^I = \sum L \times \bar{L}$ where the sum is over all irreducible representations L of $\mathfrak{G}$. Thus U^I restricted to $\tilde{\mathfrak{G}}$ is $= \sum L \otimes \bar{L}$; that is (b) and (d) are equivalent. This completes the proof of the theorem.

The equivalence of (a), (c) and (d) is established in Frame's paper ([2], Theorem 1). We remark with Frame that the equivalence of (a) and (d) furnishes a proof of the well known fact that the number of irreducible representations of a group is equal to the number of classes. Indeed in each sum every summand contains the identity exactly once.

We not also that if s is in the center of $\mathfrak{G}$ then $f(sxs^{-1}) = f(x)$ for all f and x so that the conjugating representation reduces to the identity on the center. Hence if the center of $\mathfrak{G}$ is non trivial no representation of the form $L \otimes \bar{L}$ where L is irreducible can be faithful. This is Frame's Theorem 2.

7. The infinite case. In part at least the results of the present paper may be extended to infinite dimensional unitary representations of locally compact topological groups. We have not yet actually written down the proof but there seems to be no difficulty in adapting the methods used in [5] in extending Theorem 2 of [4] to the infinite case and thus obtaining a corresponding extension of Theorem 1 of the present paper. The same remark applies to Theorem 2 and 2′ so long as we deal with strong intertwining numbers in the sense of [5] or consider only almost periodic representations. For compact groups at least then there should be no difficulty in establishing the obvious generalizations of Theorems 3, 4 and 10, the equivalence of (b) and (c) in Theorems 8 and that form of Theorem 9 in which the equation involving ζ and v are replaced by statements asserting the self inverseness of certain double cosets. For non compact groups the situation is more complicated and has not yet been thoroughly explored. We hope to give details in the compact case and do what can be done in more general cases in a subsequent paper.

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ON VALUES OMITTED BY UNIVALENT FUNCTIONS.*

By James A. Jenkins.

1. Let S denote the family of functions $f(z)$ regular and univalent for $|z| < 1$ with the expansion $f(z) = z + a_2 z^2 + \cdots$ about $z = 0$. It is well known that $f(z) \varepsilon S$ assumes for $|z| < 1$ all values w with $|w| < \frac{1}{4}$. Further it can omit just one value w with $|w| = \frac{1}{4}$. The latter can occur only for certain well known and explicitly given functions. On the other hand for $\rho \geqq 1$ there exists a function $f(z) \varepsilon S$ omitting the entire circle $|w| = \rho$ (for example $f(z) \equiv z$). It seems natural to inquire how many values can be omitted on a circle $|w| = r$, $\frac{1}{4} < r < 1$. Let $L(f, r)$ denote the length of the set of values on $|w| = r$ not covered by values of $f(z) \varepsilon S$ for $|z| < 1$. In this paper we will give the precise upper bound for $L(f, r)$ for $f \varepsilon S$ and for each value of r in $\frac{1}{4} < r < 1$. We will then apply this result to improve a result of A. W. Goodman [1].

2. We begin by constructing an explicit conformal mapping. Regard in the ζ-plane the domain D bounded by $|\zeta| = 1$ together with the portion of the real axis $1 \leqq \zeta \leqq \infty$ and distinguish the point $\zeta_0 = -\rho$ $(\rho > 1)$ within this domain. The function $\eta = \zeta + \zeta^{-1} + 2$ maps this domain on the η-plane slit along the positive real axis. ζ_0 goes into $\eta_0 = -(\rho + \rho^{-1} - 2)$. The function $W = \eta^{\frac{1}{2}}$ (taking the positive determination on the upper side of the positive real axis) maps the preceding domain on the upper half W-plane, η_0 going into $W_0 = i(\rho^{\frac{1}{2}} - \rho^{-\frac{1}{2}})$. Finally $z = -(W - W_0)/(W - \bar{W}_0)$ maps the latter domain on $|z| < 1$, W_0 going into $z = 0$. An elementary calculation shows that $d\zeta/dz\,|_{z=0} = 4\rho(\rho - 1)/(\rho + 1)$. On the other hand the function $Z = \rho(\zeta^2 + \rho\zeta)/(\rho\zeta + 1)$ maps D on a domain D^* bounded by an arc on $|Z| = \rho$ placed symmetrically with respect to the positive real axis together with the portion $\rho \leqq Z \leqq \infty$ of the latter. The point ζ_0 goes into $Z = 0$. By an elementary calculation $dZ/d\zeta\,|_{\zeta=\zeta_0} = \rho^2/(\rho^2 - 1)$. As a function of z, Z maps $|z| < 1$ on the domain D^*, and $dZ/dz\,|_{z=0} = 4\rho^3/(\rho + 1)^2$.

Thus the function $w = (\rho + 1)^2 Z / 4\rho^3$ belongs to S. It maps $|z| < 1$ on a domain $\tilde{D}(r)$ bounded by an arc on the circle $|w| = r = (\rho + 1)^2/(2\rho)^2$

* Received October 24, 1952.

placed symmetrically with respect to the positive real axis, together with the portion $r \leqq w \leqq \infty$ of the latter. As ρ takes the values in $\infty > \rho > 1$, r takes the values in $\frac{1}{4} < r < 1$.

We wish now to determine the length of the arc on $|w| = r$. The end points of the corresponding arc in the Z-plane are the images of the points on $|\zeta| = 1$ where $dZ/d\zeta = 0$ i.e. the solutions of $\rho\zeta^2 + 2\zeta + \rho = 0$. These are the points $\zeta = (-1 \pm i(\rho^2 - 1)^{\frac{1}{2}})/\rho$ and their images are $Z = (\rho^2 - 2 \pm 2i(\rho^2 - 1)^{\frac{1}{2}})/\rho$. The angle subtended at the origin by this arc is $\theta = 2\cos^{-1}(1 - 2/\rho^2)$ (the principal branch of $\cos^{-1}$ being used). Now $\rho = (2r^{\frac{1}{2}} - 1)^{-1}$ thus $\theta = 2\cos^{-1}(8r^{\frac{1}{2}} - 8r - 1)$. Hence the length of the arc is $2r\cos^{-1}(8r^{\frac{1}{2}} - 8r - 1)$. We observe this tends to 0 as r tends to $\frac{1}{4}$, and tends to 2π as r tends to 1.

3. THEOREM. *For* $\frac{1}{4} < r < 1$, $L(f, r) \leqq 2r\cos^{-1}(8r^{\frac{1}{2}} - 8r - 1)$.

For $f \varepsilon S$ let $D(f)$ be the image domain of $|z| < 1$ under f. Let $D^*(f)$ be the domain obtained from $D(f)$ by circular symmetrization in the following manner: for all s, $0 < s < \infty$ if $D(f) \cap \{|w| = s\}$ consists of the whole circumference $|w| = s$, $D^*(f) \cap \{|w| = s\}$ shall do the same; otherwise $D^*(f) \cap \{|w| = s\}$ shall consist of a single arc on $|w| = s$ of length equal to that of $D(f) \cap \{|w| = s\}$ and centred at the point $w = -s$. It is well known that we obtain in this way a domain $D^*(f)$ whose inner conformal radius with respect to $w = 0$ is at least 1. Unfortunately there seems no reference where this result is treated with complete precision. It seems to be most readily established by using the methods of Pólya and Szegö [3, 4]. A rough sketch of a proof appears in the report of Hayman [2]. For a careful treatment of similar problems by a different method see [5].

Now supose that we had $L(f, r) > 2r\cos^{-1}(8r^{\frac{1}{2}} - 8r - 1)$ for some $f \varepsilon S$ and some r, $\frac{1}{4} < r < 1$. Then $D^*(f)$ would be a proper subdomain of the domain $D(r)$ of § 2. Since the latter has inner conformal radius 1 with respect to the origin, we would be led to a contradiction. This proves the theorem.

4. A few years ago Goodman [1] proved that the greatest lower bound A of $A_f = \text{area}\, D(f) \cap \{|w| < 1\}$ for $f \varepsilon S$ satisfies $.5000\pi \leqq A < .7728\pi$. The preceding theorem enables us to improve the first bound to $.5387\pi < A$. Indeed the area of the part of $|w| < 1$ omitted by f for $|z| < 1$ cannot exceed

$$\int_{\frac{1}{4}}^{1} 2r \cos^{-1}(8r^{\frac{3}{2}} - 8r - 1)\, dr.$$

Integrating this explicitly by elementary means we find its value $.4613\pi$ to that degree of accuracy. Thus the area covered is at least $.5387\pi$. It is clear that this is still apreciably less than the true greatest lower bound.

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